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# Quantisation commutes with reduction for cocompact Hamiltonian group actions

Een wetenschappelijke proeve op het gebied van de  
Natuurwetenschappen, Wiskunde en Informatica

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# Introduction

## Historical background

In their 1982 paper [28], Guillemin and Sternberg proved a theorem that became known as ‘quantisation commutes with reduction’, or symbolically,  $[Q, R] = 0$ . For a Hamiltonian action by a compact Lie group  $K$  on a compact Kähler manifold  $(M, \omega)$ , their result asserts that the space of  $K$ -invariant vectors in the geometric quantisation space of  $(M, \omega)$  equals the geometric quantisation of the symplectic reduction of  $(M, \omega)$  by the action of  $K$ . Here geometric quantisation was defined as the (finite-dimensional) space of holomorphic sections of a certain holomorphic line bundle over  $M$ .

A more general definition of geometric quantisation, attributed to Bott, is formulated in terms of Dirac operators. A compact symplectic  $K$ -manifold  $(M, \omega)$  always admits a  $K$ -equivariant almost complex structure that is compatible with  $\omega$ , even if the manifold is not Kähler. Via this almost complex structure, one can define a Dolbeault–Dirac operator or a  $\text{Spin}^c$ -Dirac operator, coupled to a certain line bundle, whose index is interpreted as the geometric quantisation of  $(M, \omega)$ . Alternatively, one can associate a  $\text{Spin}^c$ -structure to the symplectic form  $\omega$ , and define the quantisation of  $(M, \omega)$  as the index of a  $\text{Spin}^c$ -Dirac operator on the associated spinor bundle. Since Dirac operators are elliptic, and since  $M$  is compact, these indices are well-defined formal differences of finite-dimensional representations of  $K$ , that is to say, elements of the representation ring of  $K$ .

In this more general setting, the fact that quantisation commutes with reduction, or ‘Guillemin–Sternberg conjecture’, was proved in many different ways, and in various degrees of generality, by several authors [38, 59, 60, 63, 79, 84]. The requirement that  $M$  and  $K$  are compact remained present, however. An exception is the paper [64], in which Paradan proves a version of the Guillemin–Sternberg conjecture where  $M$  is allowed to be noncompact in certain circumstances. An approach to quantising actions by noncompact groups on noncompact manifolds was also given by Vergne, in [83].

These compactness assumptions are undesirable from a physical point of view, since most classical phase spaces (such as cotangent bundles) are not compact. Furthermore, one would also like to admit noncompact symmetry groups. However, dropping the compactness assumptions poses severe mathematical difficulties, since the index of a Dirac operator on a noncompact manifold is no longer well-defined, and neither is the representation ring of a noncompact group.

In [50], Landsman proposes a solution to these problems, at least in cases where the quotient of the group action is compact. (The action is then said to be *cocompact*.) He replaces the representation ring of a group by the  $K$ -theory of its  $C^*$ -algebra, and the equivariant index by the

analytic assembly map that is used in the Baum–Connes conjecture. Landsman’s formulation of the Guillemin–Sternberg conjecture reduces to the case proved in [38, 59, 60, 63, 79, 84] if the manifold and the group in question are compact. The advantage of this formulation is that it still makes sense if one only assumes compactness of the orbit space of the action.

The first main result in this thesis is a proof of Landsman’s generalisation of the Guillemin–Sternberg conjecture for Hamiltonian actions by groups  $G$  with a normal, discrete subgroup  $\Gamma$ , such that  $G/\Gamma$  is compact.

In the compact case, the Guillemin–Sternberg conjecture implies a more general multiplicity formula for the decomposition of the geometric quantisation of  $(M, \omega)$  into irreducible representations of  $K$ . This implication is based on the Borel–Weil theorem, which is itself a special case of the multiplicity formula that follows from the Guillemin–Sternberg conjecture. In the noncompact case, it is harder to state and prove such a multiplicity formula. This is caused by the fact that the Borel–Weil theorem is a statement about *compact* groups, and by the fact that the geometric quantisation of a symplectic manifold is now a  $K$ -theory class instead of a (virtual) representation.

For semisimple groups  $G$ , we tackle these difficulties using V. Lafforgue’s work in [48] on discrete series representations and  $K$ -theory. We then obtain our second main result, which is a formula for the multiplicity of the  $K$ -theory class associated to a discrete series representation, in the geometric quantisation of a cocompact Hamiltonian  $G$ -manifold. For this result, we assume that the image of the momentum map lies in the *strongly elliptic set*. This is the set of elements of the dual of the Lie algebra of  $G$  that have compact stabilisers with respect to the coadjoint action. The coadjoint orbits in this set correspond to discrete series representations in the orbit philosophy.

## Outline of this thesis

In this thesis, we combine two branches of mathematics: symplectic geometry and noncommutative geometry. To help readers who are specialised in one of these branches understand the other one, we give a rather detailed theoretical background in Part I. In Chapter 1, which is intended for a general mathematical audience, we explain the physical motivation of the research in this thesis. Chapters 2–5 are introductions to symplectic geometry, geometric quantisation and noncommutative geometry. We conclude Part I with Chapter 6, in which we state our two main results, Theorems 6.5 and 6.13.

The proofs of these results follow the same strategy: we deduce them from the compact case of the Guillemin–Sternberg conjecture, using naturality of the assembly map. This naturality of the assembly map is the core of the noncommutative geometric part of this thesis, and is described in Part II. It contains two cases: naturality for quotient maps, and (a very special case of) naturality for inclusion maps. Besides these two cases of naturality of the assembly map, the third main result in Part II is Corollary 8.11, about the image of  $K$ -homology classes associated to elliptic differential operators under the Valette homomorphism. This homomorphism is the crucial ingredient of naturality of the assembly map for quotient maps.

In Part III, we show that the ‘Guillemin–Sternberg–Landsman’ conjecture for groups with a cocompact, normal, discrete subgroup is a consequence of Corollary 8.11. We give an alternative proof in the special case where the group is abelian and discrete, and conclude with the



example of the action of  $\mathbb{Z}^2$  on  $\mathbb{R}^2$  by addition.

To prove the multiplicity formula for discrete series representations in the case of actions by semisimple groups, we prove an intermediate result that we call ‘quantisation commutes with *induction*’. This is the central result of Part IV, and its proof is based on our version of naturality of the assembly map for inclusion maps. In this part, we define ‘Hamiltonian induction’ and ‘Hamiltonian cross-sections’, to construct new Hamiltonian actions from given ones. These constructions are each other’s inverses, and the ‘quantisation commutes with induction’-theorem provides a link between these constructions and the Dirac induction map used in the Connes–Kasparov conjecture, and (more importantly to us) in Lafforgue’s work on discrete series representations in  $K$ -theory. This will allow us to deduce the multiplicity formula for discrete series representations from the Guillemin–Sternberg conjecture in the compact case.

## Credits

Chapters 1 – 5 only contain standard material, except perhaps the alternative proof of Proposition 5.17. Section 6.1 is based on Landsman’s paper [50], and Section 6.2 is an explanation of the facts in [48] that we use. Gert Heckman proved Lemma 6.9 for us.

Chapter 7 is a reasonably straightforward generalisation of the epimorphism case of Valette’s ‘naturality of the assembly map’-result in [61] to possibly nondiscrete groups.

The idea of our proof of Theorem 6.5, as described in Section 10.1, is due to Klaas Landsman. Sections 11.1–11.3 are based on Example 3.11 from [8], and on Lusztig’s paper [55]. The proof of Lemma 11.2 was suggested to us by Elmar Schrohe.

Section 12.3 is based on the proof of the symplectic cross-section theorem in [54]. Some of the remaining facts in Chapter 12 and in Chapter 13 may be known in the case where the pair  $(G, K)$  is replaced by the pair  $(K, T)$ , although the author has not found them in the literature. The induction procedure for  $\text{Spin}^c$ -structures described in Section 13.2, was explained to us by Paul-Émile Paradan.

Our proof of Theorem 6.13 was inspired by Paradan’s article [63], and Paradan’s personal explanation of the ideas behind this paper.

## Prerequisites

This thesis is aimed at readers who are familiar with

- basic topology;
- basic Riemannian and almost complex geometry;
- basic Banach and Hilbert space theory;
- basic Lie theory, and representation theory of compact Lie groups;
- the theory of (pseudo-)differential operators on vector bundles and their principal symbols, in particular elliptic differential operators and their indices.

## Assumptions

In the topological context, all vector bundles and group actions are tacitly supposed to be continuous. In the smooth context they are supposed to be smooth.

Unless stated otherwise, all functions are complex-valued, and all Hilbert spaces and vector bundles are supposed to be complex, apart from vector bundles constructed from tangent bundles. Inner products on complex vector spaces are supposed to be linear in the first entry, and antilinear in the second one.

## Publications

Chapters 7, 8, 10 and 11 were taken from the paper [37], written jointly with Klaas Landsman, which has been accepted for publication in *K-theory*.

The end of Section 5.3, Sections 6.2 and 6.3, Chapter 9 and Chapters 12 – 15 were taken from the paper [36], which has been submitted for publication.

# **Part I**

## **Preliminaries and statement of the results**

The bulk of this first part, Chapters 2–5, consists of introductions to the two branches of mathematics that we use: symplectic geometry and noncommutative geometry. These introductions start at a basic level, so that the reader does not have to be a specialist in both of these areas to be able to read this thesis. Readers who are familiar with symplectic geometry and/or noncommutative geometry can skip the relevant chapters, or just quickly take a look at the notation and the results we will use.

In Chapter 1 we give some physical background, and in Chapter 6 we state our two main results: Theorems 6.5 and 6.13. All material in Part I is standard, except Chapter 6, and possibly the alternative proof of Proposition 5.17.

# Chapter 1

## Classical and quantum mechanics

We begin by briefly reviewing classical and quantum mechanics. This provides the physical motivation of the research in this thesis. The physical notion of quantisation will be explained, to motivate the abstract mathematical Definitions 3.15, 3.17, 3.20, 3.30 and 6.1. Chapter 1 is only meant to provide this motivation, and the rest of this thesis does not logically depend on it.

The mathematics behind classical mechanics with symmetry is treated in Chapter 2. The mathematics behind quantum mechanics with symmetry is the theory of equivariant operators on Hilbert spaces carrying unitary representations of a Lie group. Chapters 4 and 5 on noncommutative geometry deal with a way of looking at this theory.

### 1.1 Classical mechanics

Let us look at an example. Consider a point particle of mass  $m$  moving in 3-dimensional Euclidean space  $\mathbb{R}^3$ . Let  $q = (q^1, q^2, q^3)$  be the position coordinates of the particle. Suppose the particle is acted upon by an external force field  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that is determined by a potential function  $V \in C^\infty(\mathbb{R}^3)$ , by

$$F = -\text{grad} V = -\left(\frac{\partial V}{\partial q^1}, \frac{\partial V}{\partial q^2}, \frac{\partial V}{\partial q^3}\right). \quad (1.1)$$

Then the motion of the particle, as a function of time  $t$ , is given by a curve  $\gamma$  in  $\mathbb{R}^3$ , determined by the differential equation

$$F(\gamma(t)) = m\gamma''(t), \quad (1.2)$$

which is Newton's second law  $F = ma$ .

Let  $\delta(t) := m\gamma'(t)$  be the momentum of the particle at time  $t$  as it moves along the curve  $\gamma$ . Then (1.1) and (1.2) may be rewritten as

$$\begin{aligned} \gamma'(t) &= \frac{1}{m}\delta(t); \\ \delta'(t) &= -\text{grad} V(\gamma(t)). \end{aligned} \quad (1.3)$$

Given this system of equations, the particle's trajectory is determined uniquely if both its position  $q := \gamma(t_0)$  and momentum  $p := \delta(t_0)$  at a time  $t_0$  are given. This motivates the definition of the *phase space* of the particle as  $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ , consisting of all possible positions

$q = (q^1, q^2, q^3)$  and momenta  $p = (p^1, p^2, p^3)$  the particle can have. A point in phase space, called a *state*, determines the motion of the particle, through Newton's law (1.3).

To rewrite (1.3) in a way that will clarify the link between classical and quantum mechanics, consider the *Hamiltonian function*  $H \in C^\infty(\mathbb{R}^6)$ , given by the total energy of the particle:

$$H(q, p) := \frac{1}{2m} \sum_{j=1}^3 (p^j)^2 + V(q). \quad (1.4)$$

Furthermore, for two functions  $f, g \in C^\infty(\mathbb{R}^6)$ , we define the Poisson bracket

$$\{f, g\} := \sum_{j=1}^3 \frac{\partial f}{\partial p^j} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p^j} \in C^\infty(\mathbb{R}^6). \quad (1.5)$$

One can check that the Poisson bracket is a Lie bracket on  $C^\infty(\mathbb{R}^6)$ , and that it has the *derivation property* that for all  $f, g, h \in C^\infty(\mathbb{R}^6)$ ,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h. \quad (1.6)$$

The reason why we consider this bracket is that it allows us to restate (1.3) as follows. Write

$$\begin{aligned} \gamma(t) &= (\gamma^1(t), \gamma^2(t), \gamma^3(t)); \\ \delta(t) &= (\delta^1(t), \delta^2(t), \delta^3(t)). \end{aligned}$$

Then (1.3) is equivalent to the system of equations

$$\begin{aligned} (\gamma^j)'(t) &= \{H, q^j\}(\gamma(t), \delta(t)); \\ (\delta^j)'(t) &= \{H, p^j\}(\gamma(t), \delta(t)), \end{aligned} \quad (1.7)$$

for  $j = 1, 2, 3$ , where  $q^j, p^j \in C^\infty(\mathbb{R}^6)$  denote the coordinate functions. Renaming the curves  $q(t) := \gamma(t)$  and  $p(t) := \delta(t)$ , we obtain the more familiar form

$$\begin{aligned} (q^j)' &= \{H, q^j\}; \\ (p^j)' &= \{H, p^j\}. \end{aligned} \quad (1.8)$$

Here  $q^j$  and  $p^j$  denote both the components of the curves  $q$  and  $p$  and the coordinate functions on  $\mathbb{R}^6$ , making (1.8) shorter and more suggestive, but mathematically less clear than (1.7).

To describe the curves  $\gamma$  and  $\delta$  in a different way, we note that the linear map  $f \mapsto \{H, f\}$ , from  $C^\infty(\mathbb{R}^6)$  to itself, is a derivation by (1.6). Hence it defines a vector field  $\xi_H$  on  $\mathbb{R}^6$ , called the *Hamiltonian vector field* of  $H$ . Let  $e^{t\xi_H} : \mathbb{R}^6 \rightarrow \mathbb{R}^6$  be the flow of this vector field over time  $t$ . That is,

$$\left. \frac{d}{dt} \right|_{t=0} f(e^{t\xi_H}(q, p)) = \xi_H(f)(q, p) = \{H, f\}(q, p)$$

for all  $f \in C^\infty(\mathbb{R}^6)$  and  $(q, p) \in \mathbb{R}^6$ . Then, if  $\gamma(0) = q$  and  $\delta(0) = p$ , conditions (1.7) simply mean that

$$(\gamma(t), \delta(t)) = e^{t\xi_H}(q, p). \quad (1.9)$$

An *observable* in this setting is by definition a smooth function of the position and the momentum of the particle, i.e. a function  $f \in C^\infty(\mathbb{R}^6)$ . The Hamiltonian function and the Poisson bracket allow us to write the time evolution equation of any observable  $f$  as the following generalisation of (1.7):

$$\frac{d}{dt}(f(\gamma(t), \delta(t))) = \{H, f\}(\gamma(t), \delta(t)). \quad (1.10)$$

Here  $\gamma$  and  $\delta$  are curves in  $\mathbb{R}^3$  satisfying (1.7). This time evolution equation for  $f$  can be deduced from the special case (1.7) using the chain rule. We will see that (1.10) is similar to the time evolution equation (1.16) in quantum mechanics.

In (1.10), the state  $(\gamma, \delta)$  of the system changes in time, whereas the observable  $f$  is constant. To obtain a time evolution equation that resembles the quantum mechanical version more closely, we define the time-dependent version  $\tilde{f} \in C^\infty(\mathbb{R} \times \mathbb{R}^6)$  of  $f$ , by

$$\tilde{f}(t, q, p) := f(e^{t\xi_H}(q, p)) =: f_t(q, p).$$

Then by (1.9), equation (1.10) becomes

$$\left. \frac{\partial \tilde{f}}{\partial t} \right|_t = \{H, f_t\}. \quad (1.11)$$

Motivated by this example of one particle in  $\mathbb{R}^3$  moving in a conservative force field, we define a *classical mechanical system* to be a triple  $(M, \{-, -\}, H)$ , where  $M$  is a smooth manifold called the *phase space* (replacing  $\mathbb{R}^6$  in the preceding example),  $\{-, -\}$  is a Lie bracket on  $C^\infty(M)$  satisfying (1.6) for all  $f, g, h \in C^\infty(M)$ , and  $H$  is a smooth function on  $M$ , called the *Hamiltonian* function. The bracket  $\{-, -\}$  is called a Poisson bracket, and the pair  $(M, \{-, -\})$  is a *Poisson manifold*. In this thesis, we will consider *symplectic manifolds* (Definition 2.1), a special kind of Poisson manifolds. Given a classical mechanical system, the dynamics of any observable  $f \in C^\infty(M)$  is determined by the classical time evolution equation (1.11).

For more extensive treatments of the Hamiltonian formulation of classical mechanics, see [1, 2].

## 1.2 Quantum mechanics

The quantum mechanical description of a particle is quite different from the classical one. The position of a particle is no longer uniquely determined in quantum mechanics, but one can only compute the probability of finding the particle in a certain region. The same goes for any other observable.

Consider once more a particle moving in  $\mathbb{R}^3$ . The probability of finding the particle in a (measurable) region  $A \subset \mathbb{R}^3$  is then given by

$$\int_A |\psi(q)|^2 dq, \quad (1.12)$$

where  $\psi$  is the (*position*) *wave function* of the particle. For the integral (1.12) to be well-defined for all measurable  $A$ , it is necessary that  $\psi$  is an  $L^2$ -function. Furthermore, the probability that

the particle exists anywhere at all (which we assume...) is both equal to 1 and to

$$\int_{\mathbb{R}^3} |\psi(q)|^2 dq.$$

Therefore the  $L^2$ -norm of  $\psi$  equals 1. Finally, since for any real number  $\alpha$  the functions  $\psi$  and  $e^{i\alpha}\psi$  determine the same probability density  $|\psi|^2$ , the relevant phase space in quantum mechanics is

$$\{\psi \in L^2(\mathbb{R}^3); \|\psi\|_{L^2} = 1\} / \text{U}(1), \quad (1.13)$$

where  $\text{U}(1)$  acts on  $L^2(\mathbb{R}^3)$  by scalar multiplication. The quotient (1.13) is the projective space  $\mathbb{P}(L^2(\mathbb{R}^3))$ .

We will always work with the Hilbert space  $L^2(\mathbb{R}^3)$  rather than its projective space, since it is easier to work with in several respects, and since  $\mathbb{P}(L^2(\mathbb{R}^3))$  can obviously be recovered from it. The operators on  $\mathbb{P}(L^2(\mathbb{R}^3))$  that are relevant for quantum mechanics are induced by the unitary and anti-unitary operators on  $L^2(\mathbb{R}^3)$ . This is Wigner's theorem, see [76], Appendix D or [91], pp. 233-236.

We have so far considered a quantum mechanical system at a fixed point in time. In the *Schrödinger picture* of quantum dynamics, one considers time dependent wave functions  $\psi$  on  $\mathbb{R} \times \mathbb{R}^3$ , where the first factor  $\mathbb{R}$  represents time, denoted by  $t$ . As before, let  $m$  be the mass of the particle, and let  $V$  be the potential function that determines the force acting on it. The quantum mechanical time evolution of the state  $\psi$  is then determined by the *Schrödinger equation*<sup>1</sup>

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \sum_{j=1}^3 \frac{\partial^2 \psi}{(\partial q^j)^2} + V\psi, \quad (1.14)$$

where  $\hbar$  is Planck's constant divided by  $2\pi$ .

The differential operator<sup>2</sup>

$$H := -\frac{\hbar^2}{2m} \sum_{j=1}^3 \frac{\partial^2}{(\partial q^j)^2} + V$$

is called the *Hamiltonian* of this system. We see that the quantum mechanical Hamiltonian arises from the classical one (1.4) if we replace  $p^j$  by  $i\hbar \frac{\partial}{\partial q^j}$ . Historically, this was the very first step towards quantisation. By Stone's theorem (see [66], Theorem 7.17 or [68], Theorem VIII.7), equation (1.14) is equivalent to

$$\psi_t = e^{-\frac{it}{\hbar}H} \psi_0, \quad (1.15)$$

where  $\psi_t(q) := \psi(t, q)$  for all  $q \in \mathbb{R}^3$ .

In this quantum mechanical setting, an observable is a self-adjoint operator<sup>3</sup>  $a$  on  $L^2(\mathbb{R}^3)$ . The spectrum of such an operator is the set of possible values of the observable that can be

<sup>1</sup>If the function  $\psi$  is not sufficiently differentiable, then its derivatives should be interpreted in the distribution sense. On the domain on which the differential operator on the right hand side of (1.14) is self-adjoint, the time derivative of  $\psi$  is defined as the limit  $\frac{\partial \psi}{\partial t} \Big|_t := \lim_{h \rightarrow 0} \frac{\psi(t+h, -) - \psi(t, -)}{h}$ , with respect to the  $L^2$ -norm.

<sup>2</sup>This operator is not defined on all of  $L^2(\mathbb{R}^3)$ , but only on a dense subspace. It is an *unbounded* operator (see Section 4.3).

<sup>3</sup>Again, this operator may be unbounded, and need only be densely defined.



obtained in a measurement. The expectation value of a measurement of the observable  $a$  when the system is in the state  $\psi$  is given by

$$(\psi, a\psi)_{L^2} = \int_{\mathbb{R}^3} \psi(q) \overline{(a\psi)(q)} dq.$$

Up to now, we have used the Schrödinger picture of quantum dynamics, where states evolve in time, and observables remain fixed. In the Heisenberg picture, states are time independent, whereas observables vary in time. Thus, in our situation, an observable is a curve  $t \mapsto a_t$  of self-adjoint operators on  $L^2(\mathbb{R}^3)$ , such that for all states  $\psi$ ,

$$(\psi_0, a_t \psi_0)_{L^2} = (\psi_t, a_0 \psi_t)_{L^2}.$$

By (1.15), this implies that

$$a_t = e^{\frac{it}{\hbar}H} a_0 e^{-\frac{it}{\hbar}H}.$$

This, in turn, is equivalent to

$$\left. \frac{da_t}{dt} \right|_t = \frac{i}{\hbar} [H, a_t], \quad (1.16)$$

the commutator<sup>4</sup>  $Ha_t - a_tH$  of the operators  $H$  and  $a_t$ . This time evolution equation in quantum mechanics is very similar to the classical time evolution equation (1.11). This is the basis of any theory about quantising observables.

In general, a *quantum mechanical system* (in the Heisenberg picture) consists of a Hilbert space  $\mathcal{H}$  (replacing  $L^2(\mathbb{R}^3)$ ) called the *phase space*, and a self-adjoint operator<sup>5</sup>  $H$ , called the *Hamiltonian*. Observables are curves  $t \mapsto a_t$  of self-adjoint operators on  $\mathcal{H}$ , whose dependence on  $t$  is determined by (1.16).

## 1.3 Quantisation

The term ‘quantisation’ refers to any way of constructing the quantum mechanical description of a physical system from the classical mechanical description. To a classical mechanical system  $(M, \{-, -\}, H)$ , a quantisation procedure should associate a quantum mechanical system

$$Q(M, \{-, -\}, H) = (\mathcal{H}, \hat{H}) \quad (1.17)$$

(where the hat on  $H$  is used to distinguish the quantum Hamiltonian from the classical one). Such constructions go back to the pioneers of quantum mechanics (Bohr, Heisenberg, Schrödinger, Dirac). Overviews are given in [49, 51].

In addition, one would like to be able to quantise observables. Quantisation of observables is often required to be a Lie algebra homomorphism

$$(C^\infty(M), \{-, -\}) \xrightarrow{Q} (\{\text{self-adjoint operators on } \mathcal{H}\}, \frac{i}{\hbar}[-, -]) \quad (1.18)$$

---

<sup>4</sup>The definition of the commutator of two unbounded operators is actually a more delicate matter than we suggest here, but we will not go into this point.

<sup>5</sup>possibly unbounded

such that  $Q(H) = \hat{H}$ . If this quantisation map is a Lie algebra homomorphism, then by time evolution equations (1.11) and (1.16), we have

$$\left. \frac{dQ(f)_t}{dt} \right|_{t=0} = Q \left( \left. \frac{df_t}{dt} \right|_{t=0} \right),$$

for all  $f \in C^\infty(M)$ . However, we will see that quantisation of observables cannot be a Lie algebra homomorphism, if it is also required to have some additional desirable properties.

From a physical point of view, it is only required that the classical and quantum mechanical time evolution equations are related by quantisation ‘in the limit  $\hbar \rightarrow 0$ ’. That is, quantisation of observables should only be a Lie algebra homomorphism in this limit. If it is an actual Lie algebra homomorphism, this implies that the laws of quantum dynamics are the same as the laws of classical dynamics, which is obviously not the case. Nevertheless, the requirement that quantisation of observables is a Lie algebra homomorphism is often imposed in geometric quantisation, possibly because it is mathematically natural, and because it at least gives some relation between classical and quantum dynamics.

Other properties one might like to see in a quantisation procedure are the following (cf. [27], page 89).

- Let  $1_M$  be the constant function 1 on  $M$ , and let  $I_{\mathcal{H}}$  be the identity operator on  $\mathcal{H}$ . Then  $Q(1_M) = i\hbar I_{\mathcal{H}}$ .
- If a set of functions  $\{f_j\}_{j \in J}$  separates points almost everywhere on  $M$ , then the set of operators  $\{Q(f_j)\}_{j \in J}$  acts irreducibly, i.e. no nonzero proper subspace of  $\mathcal{H}$  is invariant under all  $Q(f_j)$ .

But Groenewold & van Hove’s ‘no go theorems’ [26, 82, 81] state that such a quantisation procedure does not exist. This may not be too surprising, given the highly restrictive assumption that quantisation of observables is a Lie algebra homomorphism.

There are various ways to define quantisation in such a way that as many as possible of the above requirements are satisfied, or that they are satisfied asymptotically ‘as  $\hbar$  tends to zero’. In this thesis however, we hardly pay any attention to the observable side (1.18) of geometric quantisation. Instead, we consider a mathematically rigorous approach to (1.17), based on *geometric quantisation* à la Bott. This procedure gives a way to construct the quantum mechanical phase space  $\mathcal{H}$  from the classical one  $(M, \{-, -\})$ . The prequantisation formula (see Definition 3.6) then gives a quantisation map for (some) observables, that is actually a Lie algebra homomorphism. But as we said, this will only be a side remark.

Quantising phase spaces may not seem like the most interesting part of quantisation, but it turns out that this has interesting features (especially mathematical ones), particularly in the presence of *symmetry*.

## 1.4 Symmetry and ‘quantisation commutes with reduction’

If a physical system possesses a symmetry, it can often be described in terms of a ‘smaller’ system. Replacing a system by this smaller system is called *reduction*. It is defined in a precise way for classical mechanics in Definitions 2.17 and 2.21 below. For quantum mechanics, this

notion of reduction is harder to define rigorously. The quantum reduction procedure we will work with is given by (6.3) and (6.12).

In classical mechanics, a symmetry of a system  $(M, \{-, -\}, H)$  is an action of a group  $G$  on  $M$  that leaves the bracket  $\{-, -\}$  and the function  $H$  invariant. Under certain circumstances (if the action is *Hamiltonian*, see Definition 2.6) such a symmetry allows us to define the reduced system

$$(M_G, \{-, -\}_G, H_G) = R(M, \{-, -\}, H).$$

In quantum mechanics, a symmetry of a system  $(\mathcal{H}, H)$  is a unitary representation of a group  $G$  on  $\mathcal{H}$ , such that  $H$  is a  $G$ -equivariant operator. We can then, again under favourable circumstances, define the reduced system

$$(\mathcal{H}_G, H_G) = R(\mathcal{H}, H).$$

The central motto in this thesis (and indeed, in its title) is ‘*quantisation commutes with reduction*’, or symbolically, ‘ $[Q, R] = 0$ ’. This is the equality

$$R(Q(M, \{-, -\}, H)) \cong Q(R(M, \{-, -\}, H)).$$

This equality is often expressed by commutativity (up to a suitable notion of isomorphism) of the following diagram:

$$\begin{array}{ccc} (M, \{-, -\}, H) & \xrightarrow{Q} & Q(M, \{-, -\}, H) =: (\mathcal{H}, \hat{H}) \\ \downarrow R & & \downarrow R \\ (M_G, \{-, -\}_G, H_G) & \xrightarrow{Q} & Q(M_G, \{-, -\}_G, H_G) \cong (\mathcal{H}_G, \hat{H}_G). \end{array}$$

If one only considers the phase space part of quantisation and reduction, then  $[Q, R] = 0$  has been proved for compact  $M$  and  $G$ . This is known as the *Guillemin–Sternberg conjecture* (see [28, 38, 59, 60, 63, 79, 84]). The goal of this thesis is to generalise the Guillemin–Sternberg conjecture to noncompact  $M$  and  $G$ , under the assumption that the orbit space  $M/G$  is still compact. To state and prove this generalisation, we use techniques from noncommutative geometry. We have found proofs in the case where  $G$  has a cocompact, discrete, normal subgroup (Theorem 6.5) and in the case where  $G$  is semisimple (Theorem 6.13).

The mathematics underlying classical mechanics is symplectic geometry, to which we now turn.

# Chapter 2

## Symplectic geometry

As we saw in Chapter 1, the mathematical structure of a classical phase space is that of a Poisson manifold. We will only consider particularly nice kinds of Poisson manifolds, namely *symplectic manifolds* (Definition 2.1). The ideal form of symmetry in the symplectic setting is a *Hamiltonian group action* (Definition 2.6). This involves an action of a Lie group that has an associated conserved quantity called a *momentum map*. For Hamiltonian actions, we can make the classical reduction process mentioned in Section 1.4 more precise (Definitions 2.17 and 2.21). We give many examples of Hamiltonian group actions, to give the reader a feeling for what is going on.

The proofs of most facts in this chapter and the next have been omitted, but they are usually straightforward. More information about the role of symplectic geometry in classical mechanics can be found for example in [29, 57, 75].

### 2.1 Symplectic manifolds

Let us define the special kind of Poisson manifold called symplectic manifold. A Poisson manifold is symplectic if the Poisson structure is nondegenerate in some sense (compare Theorems 2.4 and 2.5), which makes symplectic manifolds easier to work with than general Poisson manifolds.

**Definition 2.1.** A *symplectic manifold* is a pair  $(M, \omega)$ , where  $M$  is a smooth manifold and  $\omega$  is a differential form on  $M$  of degree 2, such that

1.  $\omega$  is *closed*, in the sense that  $d\omega = 0$ ;
2.  $\omega$  is *nondegenerate*, in the sense that for all  $m \in M$ , the map  $T_m M \rightarrow T_m^* M$ , given by  $v \mapsto \omega(v, -)$ , is a linear isomorphism.

Such a form  $\omega$  is called a *symplectic form*.

When explicitly verifying that a given two-form is nondegenerate, we will often use the fact that nondegeneracy of  $\omega$  is equivalent to the property that for all  $m \in M$  and all nonzero  $v \in T_m M$ , there is a  $w \in T_m M$  such that  $\omega_m(v, w) \neq 0$ .

**Example 2.2.** A *symplectic vector space* is a vector space equipped with a nondegenerate, antisymmetric bilinear form. When viewed as a differential form of degree 2, this bilinear form is a symplectic form on the given vector space.

The natural notion of isomorphism of symplectic manifolds is called *symplectomorphism*:

**Definition 2.3.** Let  $(M, \omega)$  and  $(N, \nu)$  be symplectic manifolds. A diffeomorphism  $\varphi : M \rightarrow N$  is called a *symplectomorphism* if  $\varphi^* \nu = \omega$ .

Let  $(M, \omega)$  be a symplectic manifold. The canonical Poisson bracket  $\{-, -\}$  on  $C^\infty(M)$  is defined as follows. For  $f \in C^\infty(M)$ , the *Hamiltonian vector field*  $\xi_f$  of  $f$  is defined by the equality

$$df = \omega(\xi_f, -) \quad \in \Omega^1(M). \quad (2.1)$$

Because  $\omega$  is nondegenerate, this determines  $\xi_f$  uniquely. We then set

$$\{f, g\} := \xi_f(g) = \omega(\xi_g, \xi_f) = -\xi_g(f) \quad \in C^\infty(M),$$

for  $f, g \in C^\infty(M)$ . This can be shown to be a Poisson bracket, as defined at the end of Section 1.1. In particular, the Jacobi identity for  $\{-, -\}$  follows from the fact that  $\omega$  is closed.

It follows from the nondegeneracy of  $\omega$  that  $M$  is even-dimensional. From a physical point of view, this corresponds to the fact that to each ‘position dimension’ in a classical phase space, there is an associated ‘momentum dimension’. The simplest example is the manifold  $M := \mathbb{R}^{2n}$ , for an  $n \in \mathbb{N}$ , with coordinates

$$(q, p) = (q^1, p^1, \dots, q^n, p^n),$$

and the symplectic form

$$\omega := \sum_{j=1}^n dp^j \wedge dq^j. \quad (2.2)$$

In fact, all symplectic manifolds are locally of this form:

**Theorem 2.4 (Darboux).** *Let  $(M, \omega)$  be a symplectic manifold, and let  $m \in M$  be given. Then there exists an open neighbourhood  $U \ni m$  and local coordinates  $(q, p)$  on  $U$ , such that*

$$\omega|_U = \sum_{j=1}^n dp^j \wedge dq^j.$$

The coordinates  $(q, p)$  are called *Darboux coordinates*. For a proof of this theorem, see [29], Theorem 22.1.

In Darboux coordinates, the Poisson bracket associated to the symplectic form is given by the standard expression (1.5), with 3 replaced by  $n := \dim M/2$ , and  $f, g \in C^\infty(M)$ . The difference between symplectic manifolds and general Poisson manifolds is illustrated nicely by Weinstein’s following result (see [88], Corollary 2.3).

**Theorem 2.5.** *Let  $(M, \{-, -\})$  be a Poisson manifold, and let  $m \in M$  be given. Then there exists an open neighbourhood  $U$  of  $m$  and local coordinates  $(q, p, c)$  on  $U$ , such that in these coordinates, the Poisson bracket has the standard form (1.5).*

The coordinates  $(q, p, c)$  are called *Darboux–Weinstein coordinates*. Here  $q$  and  $p$  are maps  $U \rightarrow \mathbb{R}^n$ , for the same  $n \in \mathbb{N}$ , and  $c$  is a map from  $U$  to  $\mathbb{R}^{\dim M - 2n}$ .

In the Section 2.3, we will give some more examples of symplectic manifolds. We will then also see that the natural group actions defined on these examples are in fact *Hamiltonian*.

## 2.2 Hamiltonian group actions

The relevant actions of a group  $G$  on a symplectic manifold  $(M, \omega)$  are those that leave the symplectic form  $\omega$  invariant:  $g^*\omega = \omega$  for all  $g \in G$ . Such actions are called *symplectic* actions. Suppose that  $G$  is a Lie group, and that  $(M, \omega)$  is a symplectic manifold equipped with a symplectic  $G$ -action. For every  $X \in \mathfrak{g}$  (the Lie algebra of  $G$ ), we have the induced vector field  $X_M$  on  $M$ , given by

$$(X_M)_m := X_m := \left. \frac{d}{dt} \right|_{t=0} \exp(tX)m, \quad (2.3)$$

for all  $m \in M$ . Because the action is symplectic, the Lie derivative  $\mathcal{L}_X \omega$  equals zero for each  $X \in \mathfrak{g}$ . Using Cartan's formula  $\mathcal{L}_X = di_{X_M} + i_{X_M}d$  (where  $i_{X_M}$  denotes contraction with  $X_M$ ), we get

$$0 = \mathcal{L}_X \omega = d(i_{X_M} \omega), \quad (2.4)$$

since  $d\omega = 0$ . In other words, the one-form  $i_{X_M} \omega$  is closed. The action is called *Hamiltonian* if this form is *exact*, in the following special way:

**Definition 2.6.** In the above situation, the action of  $G$  on  $(M, \omega)$  is called *Hamiltonian* if there exists a smooth map

$$\Phi : M \rightarrow \mathfrak{g}^*$$

with the following two properties.

1. For all  $X \in \mathfrak{g}$ , let  $\Phi_X \in C^\infty(M)$  be the function defined by pairing  $\Phi$  with  $X$ . Its derivative is given by

$$\boxed{d\Phi_X = -i_{X_M} \omega.} \quad (2.5)$$

2. The map  $\Phi$  is equivariant<sup>1</sup> with respect to the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

Such a map  $\Phi$  is called a *momentum map*<sup>2</sup> of the action.

Note that if  $G$  is connected, equation (2.4) implies that every Hamiltonian  $G$ -action is symplectic. Because we will also consider non-connected groups, we reserve the term Hamiltonian for symplectic actions.

Property (2.5) can be rephrased in terms of Hamiltonian vector fields, by saying that for all  $X \in \mathfrak{g}$ , one has  $\xi_{\Phi_X} = -X_M$ . If  $G$  is connected, then  $\Phi$  is equivariant if and only if for all  $X, Y \in \mathfrak{g}$ , we have  $\{\Phi_X, \Phi_Y\} = \Phi_{[X, Y]}$ . That is, if and only if  $\Phi$  is a Poisson map with respect to the standard Poisson structure on  $\mathfrak{g}^*$ .

The presence or absence of minus signs in these formulas depends on the sign conventions used in the definitions of momentum maps, Hamiltonian vector fields and vector fields induced by Lie algebra elements.

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<sup>1</sup>Sometimes a momentum map is not required to be equivariant, and the action is called *strongly Hamiltonian* if it is.

<sup>2</sup>or ‘moment map’, as people on the east coast of the United States like to say

**Remark 2.7** (Uniqueness of momentum maps). If  $\Phi$  and  $\Phi'$  are two momentum maps for the same action, then for all  $X \in \mathfrak{g}$ ,

$$d(\Phi_X - \Phi'_X) = 0.$$

If  $M$  is connected, this implies that the difference  $\Phi_X - \Phi'_X$  is a constant function, say  $c_X$ , on  $M$ . By definition of momentum maps, the constant  $c_X$  depends linearly on  $X$ . So there is an element  $\xi \in \mathfrak{g}^*$  such that

$$\Phi - \Phi' = \xi.$$

By equivariance of momentum maps, the element  $\xi$  is fixed by the coadjoint action of  $G$  on  $\mathfrak{g}^*$ . In fact, given a momentum map, the space of elements of  $\mathfrak{g}^*$  that are fixed by the coadjoint action parametrises the set of all momentum maps for the given action.

In the next section we give some examples of Hamiltonian group actions. We end this section by giving some techniques to construct new examples from given ones.

**Lemma 2.8** (Restriction to subgroups). *Let  $H < G$  be a closed subgroup, with Lie algebra  $\mathfrak{h}$ . Let*

$$p : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$$

*be the restriction map from  $\mathfrak{g}$  to  $\mathfrak{h}$ .*

*Suppose that  $G$  acts on a symplectic manifold  $(M, \omega)$  in a Hamiltonian way, with momentum map  $\Phi : M \rightarrow \mathfrak{g}^*$ . Then the restricted action of  $H$  on  $M$  is also Hamiltonian. The composition*

$$M \xrightarrow{\Phi} \mathfrak{g}^* \xrightarrow{p} \mathfrak{h}^*$$

*is a momentum map.*

**Remark 2.9.** An interpretation of Lemma 2.8 is that the momentum map is functorial with respect to symmetry breaking. For example, consider a physical system of  $N$  particles in  $\mathbb{R}^3$  (Example 2.16). If we add a function to the Hamiltonian that is invariant under orthogonal transformations, but not under translations, then the Hamiltonian is no longer invariant under the action of the Euclidean motion group  $G$ . However, it is still preserved by the subgroup  $O(3)$  of  $G$ . In other words, the  $G$ -symmetry of the system is broken into an  $O(3)$ -symmetry. By Lemma 2.8, angular momentum still defines a momentum map, so that it is still a conserved quantity (see Remark 2.15).

**Lemma 2.10** (Invariant submanifolds). *Let  $(M, \omega)$  be a symplectic manifold, equipped with a Hamiltonian action of  $G$ , with momentum map  $\Phi : M \rightarrow \mathfrak{g}^*$ . Let  $N \subset M$  be a  $G$ -invariant submanifold, with inclusion map  $j : N \hookrightarrow M$ . Assume that the restricted form  $j^*\omega$  is a symplectic form on  $N$  (i.e. that it is nondegenerate). Then the action of  $G$  on  $N$  is Hamiltonian. The composition*

$$N \xrightarrow{j} M \xrightarrow{\Phi} \mathfrak{g}^*$$

*is a momentum map.*

The next lemma will play a role in Example 2.16, and in the *shifting trick* (Remark 2.22).

Let  $(M_1, \omega_1)$  and  $(M_2, \omega_2)$  be symplectic manifolds. Suppose that there is a Hamiltonian action of a group  $G$  on both symplectic manifolds, with momentum maps  $\Phi_1$  and  $\Phi_2$ , respectively. The Cartesian product manifold  $M_1 \times M_2$  carries the symplectic form  $\omega_1 \times \omega_2$ , which is defined as

$$\omega_1 \times \omega_2 := p_1^* \omega_1 + p_2^* \omega_2,$$

where  $p_i : M_1 \times M_2 \rightarrow M_i$  denotes the canonical projection map.

Consider the diagonal action of  $G$  on  $M_1 \times M_2$ ,

$$g \cdot (m_1, m_2) = (g \cdot m_1, g \cdot m_2),$$

for  $g \in G$  and  $m_i \in M_i$ .

**Lemma 2.11** (Cartesian products). *This action is Hamiltonian, with momentum map*

$$\begin{aligned} \Phi_1 \times \Phi_2 : M_1 \times M_2 &\rightarrow \mathfrak{g}^*, \\ (\Phi_1 \times \Phi_2)(m_1, m_2) &= \Phi_1(m_1) + \Phi_2(m_2), \end{aligned}$$

for  $m_i \in M_i$ .

## 2.3 Examples of Hamiltonian actions

The most common classical phase spaces are cotangent bundles.

**Example 2.12** (Cotangent bundles). Let  $N$  be a smooth manifold, and let  $M := T^*N$  be its cotangent bundle, with projection map  $\pi_N : T^*N \rightarrow N$ . The *tautological 1-form*  $\tau$  on  $M$  is defined by

$$\langle \tau_\eta, v \rangle = \langle \eta, T_\eta \pi_N(v) \rangle,$$

for  $\eta \in T^*N$  and  $v \in T_\eta M$ . The one-form  $\tau$  is called ‘tautological’ because for all 1-forms  $\alpha$  on  $N$ , we have

$$\alpha^* \tau = \alpha.$$

Here on the left hand side,  $\alpha$  is regarded as a map from  $N$  to  $M$ , along which the form  $\tau$  is pulled back.

Let  $q = (q^1, \dots, q^d)$  be local coordinates on an open neighbourhood of an element  $n$  of  $N$ . Consider the corresponding coordinates  $p$  on  $T^*N$  in the fibre direction, defined by  $p^k = \frac{\partial}{\partial q^k}$ . Then locally, one has

$$\tau = \sum_k p^k dq^k.$$

The 2-form

$$\sigma := d\tau = \sum_k dp_k \wedge dq_k \tag{2.6}$$

is a symplectic form on  $M$ , called the *canonical symplectic form*.

Let  $G$  be a Lie group acting on  $N$ . The induced action of  $G$  on  $M$ ,

$$g \cdot \eta := (T_{gn} g^{-1})^* \eta \in T_{gn}^* N,$$



for  $g \in G$ ,  $n \in N$  and  $\eta \in T_n^*N$ , is Hamiltonian, with momentum map

$$\Phi_X = i_{X_M} \tau,$$

for all  $X \in \mathfrak{g}$ . Explicitly:

$$\Phi_X(\eta) := \langle \eta, X_{\pi_N(\eta)} \rangle,$$

for  $X \in \mathfrak{g}$  and  $\eta \in T^*N$ .

The following example forms the basis of Kirillov's 'orbit method' [42, 43, 44]. The idea behind this method is that unitary irreducible representations can sometimes be obtained as geometric quantisations of coadjoint orbits. An example of this idea is the Borel–Weil theorem (Example 3.36), which can be used to generalise the 'quantisation commutes with reduction' theorem in the compact setting (Theorem 3.34) to a statement about reduction at arbitrary irreducible representations (Theorem 3.35), as shown in Lemma 3.37.

**Example 2.13** (Coadjoint orbits). Let  $G$  be a connected Lie group. Fix an element  $\xi \in \mathfrak{g}^*$ . We define the bilinear form  $\omega^\xi$  on  $\mathfrak{g}$  by

$$\omega^\xi(X, Y) := -\langle \xi, [X, Y] \rangle,$$

for all  $X, Y \in \mathfrak{g}$ . This form is obviously antisymmetric.

The *coadjoint action*  $\text{Ad}^*$  of  $G$  on  $\mathfrak{g}^*$  is given by

$$\langle \text{Ad}^*(g)\eta, X \rangle = \langle \eta, \text{Ad}(g^{-1})X \rangle$$

for all  $g \in G$ ,  $\eta \in \mathfrak{g}^*$  and  $X \in \mathfrak{g}$ . The infinitesimal version of this action is denoted by  $\text{ad}^*$ , and defined by

$$\langle \text{ad}^*(X)\eta, Y \rangle := -\langle \eta, [X, Y] \rangle,$$

for all  $X, Y \in \mathfrak{g}$  and  $\eta \in \mathfrak{g}^*$ .

Let  $G_\xi$  be the stabiliser group of  $\xi$  with respect to the coadjoint action:

$$G_\xi := \{g \in G; \text{Ad}^*(g)\xi = \xi\}.$$

The Lie algebra  $\mathfrak{g}_\xi$  of  $G_\xi$  equals

$$\begin{aligned} \mathfrak{g}_\xi &= \{X \in \mathfrak{g}; \text{ad}^*(X)\xi = 0\} \\ &= \{X \in \mathfrak{g}; \omega^\xi(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}, \end{aligned} \tag{2.7}$$

by definition of  $\omega^\xi$ . By (2.7), the form  $\omega^\xi$  defines a symplectic form on the quotient  $\mathfrak{g}/\mathfrak{g}_\xi$ .

Let

$$\mathcal{O}^\xi := G \cdot \xi \cong G/G_\xi$$

be the coadjoint orbit through  $\xi$ . The tangent space

$$T_\xi \mathcal{O}^\xi \cong \mathfrak{g}/\mathfrak{g}_\xi$$

carries the symplectic form  $\omega^\xi$ . This form can be extended  $G$ -invariantly to a symplectic form  $\omega$  on the whole manifold  $\mathcal{O}^\xi$ . It is shown in [44], Theorem 1, that it is closed. This symplectic form is called the *canonical symplectic form* on the coadjoint orbit<sup>3</sup>  $\mathcal{O}^\xi$ .

The coadjoint action of  $G$  on  $\mathcal{O}^\xi$  is Hamiltonian. The inclusion

$$\Phi : \mathcal{O}^\xi \hookrightarrow \mathfrak{g}^*$$

is a momentum map.

The following example can be used to show that a momentum map defines a conserved quantity of a physical system.

**Example 2.14** (Time evolution). Let  $(M, \omega)$  be a symplectic manifold, and let  $H$  be a smooth function on  $M$ . If we interpret  $H$  as the Hamiltonian of some physical system on  $M$ , then we saw in (1.9) that the time evolution of the system is given by the flow  $t \mapsto e^{t\xi_H}$  of the Hamiltonian vector field  $\xi_H$  of  $H$ . If this flow is defined for all  $t \in \mathbb{R}$ , then it defines an action of the Lie group  $\mathbb{R}$  on  $M$ . This action is Hamiltonian, with momentum map  $-H : M \rightarrow \mathbb{R} \cong \mathbb{R}^*$ . In physics, it is well known that energy, given by the Hamiltonian function, is the conserved quantity associated to invariance under time evolution. The minus sign in front of  $H$  is a consequence of our sign conventions.

**Remark 2.15.** The interpretation of a momentum map as a conserved quantity arises when a Hamiltonian action of a Lie group  $G$  on a symplectic manifold  $(M, \omega)$  is given (with momentum map  $\Phi$ ), along with a  $G$ -invariant (Hamiltonian) function  $H$  on  $M$ . Then for all  $X \in \mathfrak{g}$ , the time dependence of  $\Phi_X$  is given by

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (e^{t\xi_H})^* \Phi_X &= \xi_H(\Phi_X) \\ &= \omega(\xi_{\Phi_X}, \xi_H) \\ &= -\xi_{\Phi_X}(H) \\ &= X_M(H) \\ &= 0, \end{aligned}$$

since  $H$  is  $G$ -invariant.

In terms of the Poisson bracket, the above computation shows that both time invariance of  $\Phi_X$  (for all  $X \in \mathfrak{g}$ ) and  $G$ -invariance of  $H$  (for connected  $G$ ) are equivalent to the requirement that  $\{H, \Phi_X\} = 0$  for all  $X \in \mathfrak{g}$ .

This can be seen as a form of Noether's theorem, which relates symmetries of a physical system to conserved quantities (see [27], page 16).

**Example 2.16** ( $N$  particles in  $\mathbb{R}^3$ ). To motivate the term ‘momentum map’, we give an example from classical mechanics. It is based on Example 2.12 about cotangent bundles, and Lemma 2.11 about Cartesian products.

Consider a physical system of  $N$  particles moving in  $\mathbb{R}^3$ . The corresponding phase space is the manifold

$$M := (T^*\mathbb{R}^3)^N \cong \mathbb{R}^{6N}.$$

---

<sup>3</sup>In terms of Poisson geometry, coadjoint orbits are the symplectic leaves of the Poisson manifold  $\mathfrak{g}^*$ .

Let  $(q_i, p_i)$  be the coordinates on the  $i$ th copy of  $T^*\mathbb{R}^3 \cong \mathbb{R}^6$  in  $M$ . We write

$$\begin{aligned} q_i &= (q_i^1, q_i^2, q_i^3), \\ p_i &= (p_i^1, p_i^2, p_i^3), \end{aligned}$$

and

$$(q, p) = ((q_1, p_1), \dots, (q_N, p_N)) \in M.$$

Using Example 2.12 and Lemma 2.11, we equip the manifold  $M$  with the symplectic form

$$\omega := \sum_{i=1}^N dp_i^1 \wedge dq_i^1 + dp_i^2 \wedge dq_i^2 + dp_i^3 \wedge dq_i^3.$$

Let  $G$  be the Euclidean motion group of  $\mathbb{R}^3$ :

$$G := \mathbb{R}^3 \rtimes \mathrm{O}(3),$$

whose elements are pairs  $(v, A)$ , with  $v \in \mathbb{R}^3$  and  $A \in \mathrm{O}(3)$ , with multiplication defined by

$$(v, A)(w, B) = (v + Aw, AB),$$

for all elements  $(v, A)$  and  $(w, B)$  of  $G$ . Its natural action on  $\mathbb{R}^3$  is given by

$$(v, A) \cdot x = Ax + v,$$

for  $(v, A) \in G$ ,  $x \in \mathbb{R}^3$ .

Consider the induced action of  $G$  on  $M$ . As remarked before, the physically relevant actions are those that preserve the Hamiltonian. In this example, if the Hamiltonian is preserved by  $G$  then the dynamics does not depend on the position or the orientation of the  $N$  particle system as a whole. In other words, no external forces act on the system.

By Example 2.12 and Lemma 2.11, the action of  $G$  on  $M$  is Hamiltonian. The momentum map can be written in the form

$$\Phi(q, p) = \sum_{i=1}^N (p_i, q_i \times p_i) \in (\mathbb{R}^3)^* \times \mathfrak{o}(3)^* = \mathfrak{g}^*.$$

Note that the Lie algebra  $\mathfrak{o}(3)$  is isomorphic to  $\mathbb{R}^3$ , equipped with the exterior product  $\times$ . We identify  $\mathbb{R}^3$  with its dual (and hence with  $\mathfrak{o}(3)^*$ ) via the standard inner product.

The quantity  $\sum_{i=1}^N p_i$  is the total linear momentum of the system, and  $\sum_{i=1}^N q_i \times p_i$  is the total angular momentum. As we saw in Remark 2.15, the momentum map is time-independent if the group action preserves the Hamiltonian. In this example, this implies that the total linear momentum and the total angular momentum of the system are conserved quantities.

## 2.4 Symplectic reduction

Half of the ‘quantisation commutes with reduction’ principle that is the subject of this thesis is the term ‘reduction’. Half again of this term is reduction on the classical side, which we explain in this section.

## The definition

For cotangent bundles (see Example 2.12) the appropriate notion of reduction is

$$R : T^*N \mapsto T^*(N/G), \quad (2.8)$$

which is well-defined if  $N/G$  is again a smooth manifold. Indeed,  $T^*N$  is the phase space of a system with *configuration space* (i.e. space of all possible positions)  $N$ , and it seems that  $N/G$  is a natural choice for the reduced configuration space.

More generally, we would like to associate to a Hamiltonian  $G$ -manifold  $(M, \omega)$  a symplectic manifold  $R(M, \omega)$ , in such a way that (2.8) is a special case. We immediately see that  $R(M) = M/G$  is not a good choice, since it does not generalise (2.8) unless  $G$  is discrete. Furthermore, there is no way to define a canonical symplectic form on  $M/G$  (although  $M/G$  does inherit a canonical Poisson structure from  $(M, \omega)$ ). A better definition of reduction is the following one.

**Definition 2.17.** Let  $(M, \omega)$  be a symplectic manifold, and let  $G$  be a Lie group. Suppose a Hamiltonian action of  $G$  on  $(M, \omega)$  is given, with momentum map  $\Phi$ . Suppose that  $0 \in \mathfrak{g}^*$  is a regular value<sup>4</sup> of  $\Phi$ . Then  $\Phi^{-1}(0)$  is a smooth submanifold of  $M$ , which is  $G$ -invariant by equivariance of  $\Phi$ . Suppose that the restricted action of  $G$  on  $\Phi^{-1}(0)$  is proper and free. Then the *symplectic reduction (at zero)* of the Hamiltonian action of  $G$  on  $(M, \omega)$  is the symplectic manifold  $(M_0, \omega_0)$ , where

$$M_0 := \Phi^{-1}(0)/G,$$

and  $\omega_0$  is the unique symplectic form on  $M_0$  such that

$$p^* \omega_0 = j^* \omega, \quad (2.9)$$

with  $p$  and  $j$  the quotient and inclusion maps in

$$\begin{array}{ccc} \Phi^{-1}(0) & \xhookrightarrow{j} & M \\ \downarrow p & & \\ M_0 & & \end{array}$$

**Theorem 2.18** (Marsden–Weinstein). *Such a symplectic form  $\omega_0$  exists, and is uniquely determined by the property (2.9).*

For a proof, see [58]. Another common notation for  $(M_0, \omega_0)$  is  $(M//G, \omega_{M//G})$ . Another term for symplectic reduction is *Marsden–Weinstein reduction*.

It turns out to be useful to also consider symplectic reduction at other values than  $0 \in \mathfrak{g}^*$ . Before explaining this, we look at some examples of symplectic reduction at zero.

**Proposition 2.19.** *Consider Example 2.12 about cotangent bundles. Suppose that the action of  $G$  on  $N$  is proper and free. Let  $T^*(N/G)$  be the cotangent bundle of the (smooth) quotient  $N/G$ , equipped with the canonical symplectic form  $\sigma_G = d\tau_G$ . The symplectic reduction of  $(T^*N, \sigma)$  by the action of  $G$  is symplectomorphic to  $(T^*(N/G), \sigma_G)$ :*

$$((T^*N)_0, \sigma_0) \cong (T^*(N/G), \sigma_G).$$

---

<sup>4</sup>That is, for all  $m \in \Phi^{-1}(0)$ , the tangent map  $T_m\Phi$  is surjective.

A special case of reduction of cotangent bundles is the following.

**Example 2.20** ( $N$  particles in  $\mathbb{R}^3$  revisited). In Example 2.16, we considered a classical mechanical system of  $N$  particles moving in  $\mathbb{R}^3$ . We will now describe the symplectic reduction of the phase space  $M = (T^*\mathbb{R}^3)^N$  of this system by the action of the subgroup  $\mathbb{R}^3$  of the Euclidean motion group  $G = \mathbb{R}^3 \rtimes \mathrm{O}(3)$ .

Consider the action on  $M$  of the translation subgroup  $\mathbb{R}^3$  of  $G$ . By Lemma 2.8, the total linear momentum of the system defines a momentum map for this action. By Theorem 2.19, the reduced phase space for this restricted action is

$$M_0 = (T^*\mathbb{R}^{3N})_0 = T^*(\mathbb{R}^{3N}/\mathbb{R}^3).$$

Let  $V$  be the  $(3N - 3)$ -dimensional vector space  $\mathbb{R}^{3N}/\mathbb{R}^3$ . As coordinates on  $V$ , one can take

$$\bar{q}_i := q_i - \sum_{j=1}^N c_j q_j : V \rightarrow \mathbb{R}^3, \quad i = 1, \dots, N,$$

for any set of coefficients  $\{c_j\}$  with sum 1. The coordinates then satisfy the single relation

$$\sum_{i=1}^N c_i \bar{q}_i = 0.$$

A physically natural choice for the  $c_j$  is

$$c_j := \frac{m_j}{\sum_{k=1}^N m_k},$$

where  $m_j$  is the mass of particle  $j$ . The coordinates  $\bar{q}_i$  are then related by

$$\sum_{i=1}^N m_i \bar{q}_i = 0.$$

Thus, the reduced phase space may be interpreted as the space of states of the  $N$  particle system in which the centre of mass is at rest in the origin.

## Reduction at other values of the momentum map

In the definition of symplectic reduction, we used the level set of the momentum map at the value 0. Reductions at other values also turn out to be interesting.

**Definition 2.21.** Let  $(M, \omega)$  be a symplectic manifold equipped with a Hamiltonian  $G$ -action, with momentum map  $\Phi$ . Let  $\xi \in \mathfrak{g}^*$  be given, and let  $G_\xi$  be its stabiliser with respect to the coadjoint action. Suppose that  $\xi$  is a regular value of  $\Phi$ , and that  $G_\xi$  acts properly and freely on  $\Phi^{-1}(\xi)$ . The *symplectic reduction at  $\xi$*  of the Hamiltonian action of  $G$  on  $(M, \omega)$  is then defined as the symplectic manifold  $(M_\xi, \omega_\xi)$ , where

$$M_\xi := \Phi^{-1}(\xi)/G_\xi,$$

and the symplectic form  $\omega_\xi$  is defined by the condition analogous to (2.9).

The inclusion map  $\Phi^{-1}(\xi) \hookrightarrow \Phi^{-1}(G \cdot \xi)$  induces a diffeomorphism  $M_\xi \cong \Phi^{-1}(G \cdot \xi)/G$ . When we do not specify the value at which we take a symplectic reduction, this value is always zero.

When considering questions about symplectic reductions, one can often use the *shifting trick* to generalise results about reduction at zero to results about reduction at arbitrary momentum map values.

**Remark 2.22** (The shifting trick). The symplectic reduction of a Hamiltonian group action of  $G$  on  $(M, \omega)$  at any regular value  $\xi \in \mathfrak{g}^*$  of the momentum map can be obtained as the symplectic reduction at 0 of a certain symplectic manifold containing  $M$ , by an action of  $G$ .

Indeed, let  $\mathcal{O}^\xi := G \cdot \xi \cong G/G_\xi$  be the coadjoint orbit of  $G$  through  $\xi$  (see Example 2.13). We noted that  $M_\xi \cong \Phi^{-1}(G \cdot \xi)/G$ . Consider the two symplectic manifolds  $(\mathcal{O}^{-\xi} = G \cdot (-\xi), \omega^{-\xi})$  and  $(M, \omega)$ . On these symplectic manifolds, we have Hamiltonian  $G$ -actions, with momentum maps

$$\begin{aligned} j_{-\xi} : \mathcal{O}^{-\xi} &\hookrightarrow \mathfrak{g}^* \\ \Phi : M &\rightarrow \mathfrak{g}^*. \end{aligned}$$

Consider the Hamiltonian action of  $G$  on the Cartesian product  $(\mathcal{O}^{-\xi} \times M, \omega^{-\xi} \times \omega)$  (see Lemma 2.11). As we saw, a momentum map for this action is

$$\begin{aligned} j_{-\xi} \times \Phi : \mathcal{O}^{-\xi} \times M &\rightarrow \mathfrak{g}^*, \\ (j_{-\xi} \times \Phi)(\eta, m) &:= \eta + \Phi(m), \end{aligned}$$

for  $\eta \in \mathcal{O}^{-\xi}$  and  $m \in M$ . The symplectic reduction of the action of  $G$  on  $\mathcal{O}^{-\xi} \times M$  at the value 0 is equal to the symplectic reduction of  $M$  at  $\xi$ :

$$\begin{aligned} (j_{-\xi} \times \Phi)^{-1}(0)/G &= \{(g \cdot (-\xi), m) \in \mathcal{O}^{-\xi} \times M; g \cdot (-\xi) + \Phi(m) = 0\}/G \\ &= \Phi^{-1}(G \cdot \xi)/G \\ &\cong M_\xi. \end{aligned}$$

This exhibits  $M_\xi$  as the symplectic reduction at zero of a Hamiltonian action.

The Guillemin–Sternberg conjecture, which we attempt to generalise to noncompact groups and manifolds, is usually proved for symplectic reduction at zero, and then generalised to reduction at arbitrary momentum map values via the shifting trick (see Lemma 3.37).

## Final remarks

**Remark 2.23** (Regularity assumptions). In the definition of symplectic reduction at an element  $\xi \in \mathfrak{g}^*$ , we assumed that  $\xi$  was a regular value of the momentum map  $\Phi$ , and that the stabiliser  $G_\xi$  acted properly and freely on  $\Phi^{-1}(\xi)$ . The freeness assumption may be dropped if one is willing to work with *orbifolds* instead of smooth manifolds.

Indeed, if  $\xi$  is a regular value of  $\Phi$ , then the action of  $G_\xi$  on  $\Phi^{-1}(\xi)$  is always *locally free*, i.e. has discrete stabilisers. This result is known as Smale’s lemma, see Lemma 2.24 below.

We always suppose that a given action is proper. Then all stabilisers of the action of  $G_\xi$  on  $\Phi^{-1}(\xi)$  are compact and discrete, and hence finite. This implies that for any regular value  $\xi$  of  $\Phi$ , the symplectic reduction  $M_\xi$  is an orbifold, and  $\omega_\xi$  is a symplectic form in the orbifold sense. Although we will not work with orbifolds in this thesis, we do prove our two main results in cases where the symplectic reduction is an orbifold. This is possible because the compact versions (Theorems 3.34 and 3.38) of our main results hold in the orbifold case, and because generalising these results to our noncompact settings does not require the use of orbifolds.

Worse singularities arise when  $\xi$  is not a regular value of  $\Phi$ . However, Meinrenken and Sjamaar [60] have found a way to state and prove a ‘quantisation commutes with reduction’ result in this generality, by using Kirwan’s desingularisation process [45]. Since it is not clear a priori if their approach also works for noncompact groups and manifolds, we will restrict ourselves to the orbifold case.

**Lemma 2.24** (Smale). *In the setting of Definition 2.21, the element  $\xi$  is a regular value of  $\Phi$  if and only if for all points  $m \in \Phi^{-1}(\xi)$ , the infinitesimal stabiliser  $\mathfrak{g}_m$  is trivial.*

This fact follows from the defining relation (2.5) of the momentum map. It was originally formulated in [74], Proposition 6.2.

In Part III, we will use the following ‘reduction in stages’-theorem. Let  $G$  be a Lie group, acting in Hamiltonian fashion on a symplectic manifold  $(M, \omega)$ , with momentum map  $\Phi$ . Let  $N \triangleleft G$  be a closed, normal subgroup. By Lemma 2.8, the action of  $N$  on  $M$  is Hamiltonian. Suppose that  $0 \in \mathfrak{n}^*$  is a regular value of the momentum map induced by  $\Phi$ , and let  $(M//N, \omega_{M//N})$  be the symplectic reduction at zero of this action.

**Theorem 2.25** (Reduction in stages). *The action of the quotient group  $G/N$  on  $(M//N, \omega_{M//N})$  is Hamiltonian, with momentum map  $\Phi_N : M//N \rightarrow (\mathfrak{g}/\mathfrak{n})^*$  given by*

$$\langle \Phi_N(Nm), X + \mathfrak{n} \rangle := \langle \Phi(m), X \rangle$$

*for all  $m \in M$  and  $X \in \mathfrak{g}$ . Suppose that  $0 \in \mathfrak{g}^*$  and  $0 \in (\mathfrak{g}/\mathfrak{n})^*$  are regular values of  $\Phi$  and  $\Phi_N$ , respectively. Then the symplectic reduction (at zero) of this action is symplectomorphic to the symplectic reduction  $(M//G, \omega_{M//G})$  of  $(M, \omega)$  by  $G$ .*

For a proof, see [56], or [49], Theorem IV.1.8.2.

# Chapter 3

## Geometric quantisation

This chapter is about geometric quantisation in the compact case. Some parts of it are necessary to understand Definitions 6.1 and 6.2 in the noncompact case, while other parts only serve as motivation for these definitions.

The quantisation of a symplectic manifold  $(M, \omega)$  should be a Hilbert space  $\mathcal{H}$ . The easiest way to construct such a Hilbert space would be setting

$$\mathcal{H} := L^2(M),$$

with respect to the *Liouville measure* given by the volume form  $\frac{\omega^n}{n!}$ , with  $\dim M = 2n$ . This first guess can be improved in two ways.

First of all, instead of functions on  $M$ , we will look at sections of a line bundle  $L^\omega \rightarrow M$ . Given a suitable Hermitian metric and a connection on  $L^\omega$ , we then have a way to ‘quantise observables’ (see Definition 3.6). Such a line bundle with a metric and a connection is called a *prequantisation* of  $(M, \omega)$ . This is explained in Section 3.1.

More importantly, as we saw in Section 1.2, the quantisation of  $\mathbb{R}^6$  should be  $L^2(\mathbb{R}^3)$ , not  $L^2(\mathbb{R}^6)$ . The problem how to ‘shrink’  $L^2(M, L^\omega)$  to a more appropriate quantisation space can be solved using either *polarisations* (Section 3.2) or *Dirac operators* (Sections 3.3 and 3.4).

Another indication that  $L^2(M, L^\omega)$  is ‘too big’ is that quantisation only commutes with reduction if it is defined as the smaller space mentioned in the previous paragraph. The author views the ‘quantisation commutes with reduction’ principle as an *axiom* of quantisation and reduction; if this principle is violated, then something must be wrong with the quantisation and/or reduction procedures one is using. The ‘quantisation commutes with reduction’ principle is explained in Section 3.7 for actions of compact groups on compact manifolds, and for cocompact actions it is explained in Chapter 6.

### 3.1 Prequantisation

The first step towards geometric quantisation is *prequantisation*. A prequantisation of a symplectic manifold  $(M, \omega)$  is a Hermitian line bundle  $L^\omega$  over  $M$ , equipped with a Hermitian connection whose curvature form is  $2\pi i \omega$ . The geometric quantisation of  $(M, \omega)$  will (initially) be defined as a subspace of the space of sections of this line bundle. The Hermitian structure on  $L^\omega$  turns this space into a Hilbert space, and the connection on  $L^\omega$  allows us to quantise observables to a certain extent.



## Line bundles

We begin with some background information about line bundles. Let  $M$  be a smooth manifold, and let  $L \rightarrow M$  be a smooth complex line bundle over  $M$ . The space of smooth sections of  $L$  is denoted by  $\Gamma^\infty(M, L)$ , or by  $\Gamma^\infty(L)$ . The space of smooth differential forms on  $M$  of degree  $k$ , with coefficients in  $L$ , is the space

$$\Omega^k(M; L) := \Gamma^\infty(M, \wedge^k T^*M \otimes L).$$

**Definition 3.1.** If  $(-, -)_L$  is a Hermitian metric on  $L$ , then a connection  $\nabla$  on  $L$  is called *Hermitian* if for all  $s, t \in \Gamma^\infty(M, L)$ ,

$$d(s, t)_L = (\nabla s, t)_L + (s, \nabla t)_L \in \Omega^1(M).$$

A connection  $\nabla$  on  $L$  can be uniquely extended to a linear map

$$\nabla : \Omega^k(M; L) \rightarrow \Omega^{k+1}(M; L),$$

such that for all  $\alpha \in \Omega^k(M)$  and  $\beta \in \Omega(M; L)$ , the following generalised Leibniz rule holds:

$$\nabla(\alpha \wedge \beta) = \alpha \wedge \nabla \beta + (-1)^k d\alpha \wedge \beta.$$

A consequence of this Leibniz rule is that the square of  $\nabla$ ,

$$\nabla^2 : \Omega^k(M; L) \rightarrow \Omega^{k+2}(M; L),$$

is a  $C^\infty(M)$ -linear mapping. Hence it is given by multiplication by a certain two-form.

**Definition 3.2.** The *curvature (form)* of a connection  $\nabla$  on  $L$  is the two-form

$$2\pi i \omega \in \Omega_{\mathbb{C}}^2(M) := \Gamma^\infty(M, \wedge^2 T^*M \otimes \mathbb{C})$$

such that for all  $s \in \Gamma^\infty(M, L)$ ,

$$\nabla^2 s = 2\pi i \omega \otimes s. \quad (3.1)$$

An equivalent formulation of (3.1) is that for all vector fields  $v$  and  $w$  on  $M$ , the  $C^\infty(M)$ -linear map

$$[\nabla_v, \nabla_w] - \nabla_{[v, w]} : \Gamma^\infty(M, L) \rightarrow \Gamma^\infty(M, L) \quad (3.2)$$

is given by multiplication by the function  $2\pi i \omega(v, w)$ .

It turns out that  $\omega$  is real, closed (the *Bianchi identity*), and that the cohomology class  $[\omega] \in H_{\text{dR}}^2(M)$  is *integral*. That is, it lies in the image of the map  $H^2(M; \mathbb{Z}) \rightarrow H_{\text{dR}}^2(M)$ . Or, equivalently, for all compact, two-dimensional submanifolds  $S \subset M$ , the number  $\int_S \omega$  is an integer.

Conversely, we have the following theorem. For a proof, see [93].

**Theorem 3.3 (Weil).** *Let  $M$  be a smooth manifold,  $\omega$  a real, closed two-form on  $M$ , with integral cohomology class  $[\omega] \in H_{\text{dR}}^2(M)$ .*

*Then there is a line bundle  $L^\omega \rightarrow M$ , with a Hermitian metric  $(-, -)_{L^\omega}$ , and a Hermitian connection  $\nabla$  whose curvature form is  $2\pi i \omega$ .*

**Definition 3.4.** A triple  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  as in Theorem 3.3 is a *prequantisation* for  $(M, \omega)$ . The line bundle  $L^\omega$  is called a *prequantum line bundle*.

## Observables

In this thesis, we are not concerned with quantising observables. However, to motivate the definition of prequantisation, let us explain a possible approach to quantising observables using a prequantisation. First, recall the definition (2.1) of Hamiltonian vector fields. The map  $f \mapsto \xi_f$  is a Lie algebra homomorphism from the Poisson algebra  $(C^\infty(M), \{-, -\})$  of  $(M, \omega)$  to the Lie algebra  $\mathfrak{X}(M)$  of vector fields on  $M$ :

**Lemma 3.5.** *For all  $f, g \in C^\infty(M)$ ,*

$$[\xi_f, \xi_g] = \xi_{\{f, g\}}.$$

This lemma can be proved via a straightforward local verification in Darboux coordinates.

We mentioned in Section 1.3 that it is a common assumption that quantisation of observables is a Lie algebra homomorphism from the Poisson algebra  $(C^\infty(M), \{-, -\})$  to the algebra of operators on the quantum phase space, with the Lie bracket defined as the commutator. Here we omit the constant  $\frac{i}{\hbar}$  in (1.18). The quantum phase space obtained via geometric quantisation will be a subspace of the space of smooth sections of a prequantum line bundle  $L^\omega \rightarrow M$ . If the prequantisation operator (defined below) associated to a classical observable preserves this subspace, then the induced operator on the quantum phase space can be interpreted as the quantisation of the classical observable.

**Definition 3.6.** Let  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  be a prequantisation for  $(M, \omega)$ . Let  $f \in C^\infty(M)$ , and consider the linear operator  $P(f)$  on  $\Gamma^\infty(M, L^\omega)$ , defined by

$$P(f) := \nabla_{\xi_f} - 2\pi i f. \quad (3.3)$$

It is called the *prequantisation operator* of the function  $f$ .

The linear map

$$P : C^\infty(M) \rightarrow \text{End}(\Gamma^\infty(M, L^\omega))$$

defined by (3.3), is called *prequantisation*.

Prequantisation is indeed a Lie algebra homomorphism:

**Theorem 3.7** (Kostant – Souriau). *Prequantisation is a Lie algebra homomorphism with respect to the Poisson bracket on  $C^\infty(M)$  and the commutator bracket of operators on  $\Gamma^\infty(M, L^\omega)$ .*

A proof of this theorem can be given by using Lemma 3.5 and the fact that  $\nabla^2 = 2\pi i \omega$ . This is a reason for looking at sections of a prequantum bundle instead of at functions.

## Equivariant prequantisations

Since we are interested in Hamiltonian group actions on symplectic manifolds, and not just in the symplectic manifolds themselves, we now take a look at prequantisations of such group actions. Let  $(M, \omega)$  be a symplectic manifold, and let  $G$  be a Lie group acting symplectically on  $(M, \omega)$ .

**Definition 3.8.** An *equivariant prequantisation* of the action of  $G$  on  $M$  is a prequantisation  $(L^\omega(M, L^\omega), (-, -)_{L^\omega}, \nabla)$  of  $(M, \omega)$  with the following additional properties.

- $L^\omega$  is a  $G$ -equivariant line bundle;
- the metric  $(-, -)_{L^\omega}$  is  $G$ -invariant;
- the connection  $\nabla$  is  $G$ -equivariant as an operator on  $\Omega^*(M; L^\omega)$ .

Equivariance of  $\nabla$  is equivalent to the requirement that for all sections  $s \in \Gamma^\infty(L^\omega)$ , all vector fields  $v \in \mathfrak{X}(M)$  and all  $g \in G$ , we have

$$g \cdot (\nabla_v s) = \nabla_{g \cdot v} g \cdot s.$$

Here the section  $g \cdot s$  and the vector field  $g \cdot v$  are defined by

$$\begin{aligned} (g \cdot s)(m) &= g \cdot s(g^{-1}m); \\ (g \cdot v)_m &= T_{g^{-1}m}g(v_{g^{-1}m}). \end{aligned} \tag{3.4}$$

for all  $m \in M$ .

**Remark 3.9** (Existence of equivariant prequantisations). As can be seen in the example in Section 11.5, it is not always clear if an equivariant prequantisation exists.

If  $G$  is compact, then existence of an equivariant prequantisation is equivalent to integrality of the equivariant cohomology class  $[\omega - \Phi]$  (see [27], Theorem 6.7). If the manifold  $M$  is simply connected and the group  $G$  is discrete, then Hawkins [32] gives a procedure to lift the action of  $G$  on  $M$  to a projective action on the trivial line bundle over  $M$ , such that a given connection is equivariant. Under a certain condition (integrality of a group cocycle), this projective action is an actual action.

In general however, existence of an equivariant prequantisation of a given Hamiltonian action does not follow from a result like Theorem 3.3, and has to be assumed. In Section 13.1, we show how in some cases, an equivariant prequantisation can be constructed from a prequantisation of an action by a compact group on a compact submanifold.

In the literature on the Guillemin–Sternberg conjecture, usually a more specific kind of equivariant prequantisation is considered. To define this prequantisation, suppose that  $(M, \omega)$  is a Hamiltonian  $G$ -manifold, with momentum map  $\Phi$ . Let  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  be a prequantisation of  $(M, \omega)$ , which is not yet assumed to be equivariant. Suppose  $L^\omega$  is a  $G$ -line bundle. The induced action of the Lie algebra  $\mathfrak{g}$  on  $\Gamma^\infty(L^\omega)$  is defined by

$$X(s)(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(tX)s(\exp(-tX)m),$$

for  $X \in \mathfrak{g}$ ,  $s \in \Gamma^\infty(L^\omega)$  and  $m \in M$ .

**Proposition 3.10.** *Suppose that  $G$  is connected, and that the action of  $\mathfrak{g}$  on  $\Gamma^\infty(L^\omega)$  is given by the Kostant formula*

$$X(s) = -P(\Phi_X) = -\nabla_{\xi_{X_M}} s + 2\pi i \Phi_X s.$$

*Then  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  is an equivariant prequantisation of the action of  $G$  on  $(M, \omega)$ . That is, the metric  $(-, -)_{L^\omega}$  is  $G$ -invariant, and the connection  $\nabla$  is  $G$ -equivariant.*

The author is not aware of a proof of this fact in the literature, but such a proof is a straightforward matter of verifying the desired properties, using the fact that  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  is a prequantisation.

A reason why we consider the more general equivariant prequantisations, as in Definition 3.8, is that we will also consider non-connected groups in Part III.

## 3.2 Quantisation via polarisations

The first way to quantise a prequantised symplectic manifold  $(M, \omega)$  is by using a *polarisation* of the complex tangent space  $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ .

**Definition 3.11.** Let  $(V, \omega)$  be a symplectic vector space of dimension  $2n$ . The symplectic form  $\omega$  extends complex-linearly to the complexification  $V \otimes \mathbb{C}$ . A *polarisation* of  $V \otimes \mathbb{C}$  is a complex Lagrangian subspace  $P$  of  $V \otimes \mathbb{C}$ . That is,  $P^{\perp} = P$ , where  $P^{\perp}$  is the subspace of  $V \otimes \mathbb{C}$  orthogonal to  $P$  with respect to  $\omega$ .

**Definition 3.12.** Let  $(M, \omega)$  be a symplectic manifold, and let  $P$  be a smooth subbundle of the complexified tangent bundle  $TM \otimes \mathbb{C}$ . Then  $P$  is called a *polarisation* of  $(M, \omega)$  if it has the following properties.

1. The subspace  $P_m \subset T_m M \otimes \mathbb{C}$  is a polarisation of  $(T_m M \otimes \mathbb{C}, \omega_m)$  for all  $m \in M$ .
2. The signatures  $(r_m, s_m)$  of the forms  $(-, -)_{P_m}$  on  $P_m / (P_m \cap \bar{P}_m)$  are locally constant on  $M$ .
3. The subbundle  $P$  of  $TM \otimes \mathbb{C}$  is *integrable*. That is, the space of sections of  $P$  is closed under the Lie bracket of vector fields.

**Example 3.13** (Vertical polarisation). Let  $N$  be a manifold, and let  $M$  be the cotangent bundle  $T^*N$ , equipped with the standard symplectic form  $\sigma = d\tau$  from Example 2.12. Let  $P \subset TM \otimes \mathbb{C}$  be the subbundle

$$P := \ker T_{\mathbb{C}} \pi_N,$$

where  $\pi_N : T^*N \rightarrow N$  denotes the cotangent bundle projection. Then  $P$  is a polarisation of  $(M, \sigma)$ , called the *vertical* polarisation. Note that

$$P \cong TN \otimes \mathbb{C} \hookrightarrow TM \otimes \mathbb{C}.$$

**Example 3.14** (Kähler polarisation). Let  $M$  be a complex manifold, and let  $H$  be a Hermitian metric on  $TM$ . Let  $g$  be the real part of  $H$ , and let  $\omega$  be *minus* the imaginary part of  $H$ . (The minus sign makes the notation in this example compatible with the notation in the rest of this thesis.) The pair  $(M, H)$  is called a *Kähler manifold* if  $d\omega = 0$ . In that case,  $(M, \omega)$  is a symplectic manifold.

Let  $J : TM \rightarrow TM$  be the complex structure on  $M$ . Then

$$g(-, -) = \omega(-, J-)$$

is a Riemannian metric on  $M$ . Because  $g$  and  $H$  are determined by  $\omega$  and  $J$ , we may also denote the Kähler manifold  $(M, H)$  by  $(M, \omega, J)$ , or  $(M, \omega)$  by abuse of notation.

The *Kähler polarisation* of  $(M, \omega)$  is the  $-i$  eigenspace of  $J$  acting on  $TM \otimes \mathbb{C}$ :

$$P := \{JX - iX; X \in TM\}.$$

A function  $f \in C^{\infty}(M)$  is holomorphic if and only if  $Z(f) = 0$  for all  $Z \in \Gamma^{\infty}(M, P)$ .

Given a symplectic manifold  $(M, \omega)$ , a prequantisation  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  of  $(M, \omega)$  and a polarisation  $P \subset TM \otimes \mathbb{C}$ , the geometric quantisation of  $(M, \omega)$  can be defined as

$$Q_I(M, \omega) := \{s \in \Gamma^\infty(M, L^\omega); \nabla_Z s = 0 \text{ for all } Z \in \Gamma^\infty(M, P)\}. \quad (3.5)$$

This definition of quantisation is often applied to compact Kähler manifolds, and it is this case that we will generalise in the course of this chapter.

**Definition 3.15** (Quantisation I). Let  $(M, \omega)$  be a compact Kähler manifold, such that  $[\omega]$  is an integral cohomology class. Let  $P$  be the Kähler polarisation of  $M$ , and let  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  be a prequantisation. Then the *Kähler-quantisation* of  $(M, \omega)$  is the finite-dimensional vector space (3.5).

We can give the line bundle  $L^\omega$  the structure of a *holomorphic* line bundle, by requiring that its space of holomorphic sections is  $Q_I(M, \omega)$ . The vector space  $Q_I(M, \omega)$  is therefore indeed finite-dimensional. A reason for using sections of a line bundle instead of functions on  $M$  in the definition of quantisation, is the fact that there are no nonconstant holomorphic functions on a compact complex manifold, whereas a holomorphic line bundle on such a manifold may have interesting sections.

**Remark 3.16.** In the situation of Definition 3.15, consider the Dolbeault complex on  $M$  with coefficients in  $L^\omega$ :

$$0 \longrightarrow \Omega^{0,0}(M; L^\omega) \xrightarrow{\bar{\partial} \otimes 1_{L^\omega}} \Omega^{0,1}(M; L^\omega) \xrightarrow{\bar{\partial} \otimes 1_{L^\omega}} \dots \xrightarrow{\bar{\partial} \otimes 1_{L^\omega}} \Omega^{0,d_M}(M; L^\omega) \longrightarrow 0.$$

Here  $d_M$  is the real dimension of  $M$ . The zeroth cohomology space  $H^{0,0}(M; L^\omega)$  is the space of holomorphic sections of  $L^\omega$ , which we defined to be  $Q_I(M, \omega)$ . This implies that  $Q_I(M, \omega)$  is not the zero space if the line bundle  $L^\omega$  is sufficiently positive.

Indeed, if  $L^\omega \otimes \wedge^{0,d_M} TM$  is a positive line bundle, then by Kodaira’s vanishing theorem (see e.g. [90], Section VI.2), all Dolbeault cohomology spaces  $H^{0,k}(M; L^\omega)$  vanish for  $k > 0$ . The Hirzebruch–Riemann–Roch theorem expresses the number

$$\sum_{k=0}^{d_M} (-1)^k \dim H^{0,k}(M; L^\omega) = \dim H^{0,0}(M; L^\omega)$$

as the integral over  $M$  of a certain differential form. If  $L^\omega$  is positive enough, this number turns out to be nonzero.

If the line bundle  $L^\omega$  is positive, but not positive enough, then we can replace  $L^\omega$  by a tensor power  $L^{\omega \otimes n}$ , to make it sufficiently positive. This amounts to replacing the symplectic form  $\omega$  by a multiple  $n\omega$ . Roughly speaking, we can think of  $n$  as being proportional to  $1/\hbar$  so that choosing  $L^\omega$  positive enough, i.e. choosing  $n$  big enough, comes down to  $\hbar$  being small enough.

### 3.3 Quantisation via the Dolbeault–Dirac operator

In this section, we improve Definition 3.15 of geometric quantisation in two ways. First, we give a definition (Definition 3.17) that yields a nonzero quantisation in more cases than Definition

3.15, and then we rephrase Definition 3.17 in a way that allows us to generalise it to possibly non-Kähler symplectic manifolds. Both definitions reduce to Definition 3.15 if the prequantum line bundle is positive enough.

**Definition 3.17** (Quantisation II). Let  $(M, \omega)$  be a compact Kähler manifold, suppose that  $[\omega]$  is an integral cohomology class, and let  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  be a prequantisation. We define the quantisation of  $(M, \omega)$  as

$$Q_{II}(M, \omega) := \sum_{k=0}^n (-1)^k H^{0,k}(M; L^\omega),$$

the alternating sum of the Dolbeault cohomology spaces of  $M$  with coefficients in  $L^\omega$ . This is a virtual vector space, i.e. a formal difference of vector spaces, whose isomorphism class is determined by the integer

$$\sum_{k=0}^n (-1)^k \dim H^{0,k}(M; L^\omega).$$

If the line bundle  $L^\omega$  is positive enough, then the definition of quantisation agrees with the previous one (see Remark 3.16).

### The Dolbeault–Dirac operator

Definition 3.17 may be reformulated in a way that makes sense even when the manifold  $M$  is not Kähler. Let  $(M, \omega)$  be a compact symplectic manifold. Suppose that  $[\omega]$  is an integral cohomology class, and let  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  be a prequantisation. Let  $J$  be an almost complex structure on  $TM$  that is *compatible* with  $\omega$ :

**Definition 3.18.** An almost complex structure  $J$  on a symplectic manifold  $(M, \omega)$  is said to be *compatible* with  $\omega$ , if the symmetric bilinear form

$$g := \omega(-, J-)$$

is a Riemannian metric on  $M$ .

Compatible almost complex structures always exist (see for example [27], pp. 111–112).

As we noted before, the connection  $\nabla$  on  $L^\omega$  defines a differential operator

$$\nabla : \Omega^k(M; L^\omega) \rightarrow \Omega^{k+1}(M; L^\omega),$$

such that for all  $\alpha \in \Omega^k(M)$  and  $s \in \Gamma^\infty(M, L^\omega)$ ,

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s.$$

Consider the projection

$$\pi^{0,*} : \Omega_{\mathbb{C}}^*(M; L^\omega) \rightarrow \Omega^{0,*}(M; L^\omega),$$

according to the decomposition  $\Omega_{\mathbb{C}}^k(M; L^\omega) = \bigoplus_{p+q=k} \Omega^{p,q}(M; L^\omega)$ . Define the differential operator

$$\bar{\partial}_{L^\omega} : \Omega^{0,q}(M; L^\omega) \rightarrow \Omega^{0,q+1}(M; L^\omega)$$

by

$$\bar{\partial}_{L^\omega} := \pi^{0,*} \circ \nabla.$$

The Riemannian metric  $g$  induces a metric on the bundle  $\bigwedge^{0,*} T^*M$ , which we also denote by  $g$ . Let  $(-, -)$  be the inner product on  $\Omega_c^{0,*}(M; L^\omega)$  such that for all  $\alpha, \beta \in \Omega_c^{0,*}(M)$  and all  $s, t \in \Gamma^\infty(M, L^\omega)$ ,

$$(\alpha \otimes s, \beta \otimes t) = \int_M g(\alpha, \beta)(m)(s, t)_{L^\omega}(m) dm.$$

where  $dm$  is the Liouville measure. Let  $\bar{\partial}_{L^\omega}^*$  be the formal adjoint of  $\bar{\partial}_{L^\omega}$ , defined by the requirement that

$$(\bar{\partial}_{L^\omega} \varphi, \psi) = (\varphi, \bar{\partial}_{L^\omega}^* \psi)$$

for all  $\varphi, \psi \in \Omega^{0,*}(M; L^\omega)$ , where  $\varphi$  has compact support.

**Definition 3.19.** The *Dolbeault–Dirac operator* is the elliptic differential operator

$$\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^* : \Omega^{0,*}(M; L^\omega) \rightarrow \Omega^{0,*}(M; L^\omega).$$

This operator maps forms of even degree to forms of odd degree, and vice versa.

## Dolbeault-quantisation

**Definition 3.20** (Quantisation III). The *Dolbeault-quantisation* of  $(M, \omega)$  is defined as the virtual vector space

$$\ker \left( (\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)|_{\Omega^{0,\text{even}}(M; L^\omega)} \right) - \ker \left( (\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)|_{\Omega^{0,\text{odd}}(M; L^\omega)} \right),$$

which is the index of the Dolbeault–Dirac operator

$$\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^* : \Omega^{0,\text{even}}(M; L^\omega) \rightarrow \Omega^{0,\text{odd}}(M; L^\omega). \quad (3.6)$$

In other words,

$$\boxed{Q_{III}(M, \omega) := \text{index}(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)}. \quad (3.7)$$

Because this operator is elliptic and  $M$  is compact, its index is well-defined.

**Remark 3.21.** In general, let  $E = E^0 \oplus E^1 \rightarrow M$  be a  $\mathbb{Z}_2$ -graded vector bundle, equipped with a metric, over a compact manifold. Let  $D$  be an elliptic differential operator on  $E$ . Suppose that  $D$  is symmetric with respect to the  $L^2$ -inner product in sections of  $E$  with respect to a given measure on  $M$ , and that it interchanges sections of  $E^0$  and  $E^1$ . Then, as in (3.7), we will often slightly abuse notation by writing

$$\begin{aligned} \text{index } D &:= \text{index}(D : \Gamma^\infty(E^0) \rightarrow \Gamma^\infty(E^1)) \\ &= [\ker D \cap \Gamma^\infty(E^0)] - [\ker D \cap \Gamma^\infty(E^1)]. \end{aligned}$$

**Remark 3.22** (Quantisation III for Kähler manifolds). If  $M$  is a complex manifold, and  $L^\omega$  is a holomorphic line bundle over  $M$ , then we can define the elliptic differential operator

$$(\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega} : \Omega^{0,*}(M; L^\omega) \rightarrow \Omega^{0,*}(M; L^\omega) \quad (3.8)$$

as follows. Locally, one has

$$\Omega^{0,q}(U; L^\omega|_U) \cong \Omega^{0,q}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U, L^\omega|_U).$$

Here  $U$  is an open subset of  $M$  over which  $L^\omega$  trivialises,  $\mathcal{O}(U)$  denotes the space of holomorphic functions on  $U$ , and  $\mathcal{O}(U, L^\omega|_U)$  is the space of holomorphic sections of  $L^\omega$  on  $U$ . Because (by definition)  $\bar{\partial}f = 0$  for holomorphic functions  $f$ , we can locally define the differential operator

$$\bar{\partial} \otimes 1_{L^\omega} : \Omega^{0,q}(U; L^\omega|_U) \rightarrow \Omega^{0,q+1}(U; L^\omega|_U),$$

by

$$\bar{\partial} \otimes 1_{L^\omega}(\alpha \otimes s) = \bar{\partial}\alpha \otimes s,$$

for all  $\alpha \in \Omega^{0,q}(U)$  and  $s \in \mathcal{O}(U, L^\omega|_U)$ . These local operators patch together to a globally defined operator

$$\bar{\partial} \otimes 1_{L^\omega} : \Omega^{0,q}(M; L^\omega) \rightarrow \Omega^{0,q+1}(M; L^\omega),$$

from which we can define the operator (3.8) by

$$(\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega} := \bar{\partial} \otimes 1_{L^\omega} + (\bar{\partial} \otimes 1_{L^\omega})^*.$$

If  $(M, \omega)$  is a compact Kähler manifold that admits a prequantum line bundle  $(L^\omega, (-, -)_{L^\omega}, \nabla)$ , then the Dolbeault–Dirac operator  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  turns out to have the same principal symbol, and hence the same index, as the operator  $(\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega}$ . So for Kähler manifolds, Definition 3.20 may be rephrased as

$$Q_{III}(M, \omega) := \text{index} \left( (\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega} : \Omega^{0,\text{even}}(M; L^\omega) \rightarrow \Omega^{0,\text{odd}}(M; L^\omega) \right).$$

**Lemma 3.23.** *If  $(M, \omega)$  is a Kähler manifold, then Definitions II and III of geometric quantisation agree.*

*Proof.* Note that

$$\begin{aligned} H^{0,k}(M; L^\omega) &= \ker \left( \bar{\partial}^k \otimes 1_{L^\omega} \right) / \text{im} \left( \bar{\partial}^{k-1} \otimes 1_{L^\omega} \right) \\ &\cong \ker \left( \bar{\partial}^k \otimes 1_{L^\omega} \right) \cap \left( \text{im} \left( \bar{\partial}^{k-1} \otimes 1_{L^\omega} \right) \right)^\perp \\ &= \ker \left( \bar{\partial}^k \otimes 1_{L^\omega} \right) \cap \ker \left( \bar{\partial}^{k-1} \otimes 1_{L^\omega} \right)^* \\ &= \ker \left( \left( \bar{\partial}^k + \left( \bar{\partial}^{k-1} \right)^* \right) \otimes 1_{L^\omega} \right), \end{aligned}$$

because the images of  $\bar{\partial}^k$  and  $(\bar{\partial}^{k-1})^*$  lie in different spaces.



We conclude that

$$\begin{aligned} H^{0,\text{even}}(M; L^\omega) &= \bigoplus_{k \text{ even}} \ker \left( \bar{\partial}^k + \left( \bar{\partial}^{k-1} \right)^* \right) \otimes 1_{L^\omega} \\ &= \ker \left( (\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega} |_{\Omega^{0,\text{even}}(M; L^\omega)} \right), \end{aligned}$$

and similarly,

$$\begin{aligned} H^{0,\text{odd}}(M; L^\omega) &= \bigoplus_{k \text{ odd}} \ker \left( \bar{\partial}^k + \left( \bar{\partial}^{k-1} \right)^* \right) \otimes 1_{L^\omega} \\ &= \ker \left( (\bar{\partial} + \bar{\partial}^*) \otimes 1_{L^\omega} |_{\Omega^{0,\text{odd}}(M; L^\omega)} \right). \end{aligned}$$

□

### 3.4 Quantisation via the $\text{Spin}^c$ -Dirac operator

Prequantisations and almost complex structures are the crucial ingredients of the definition of quantisation via the Dolbeault–Dirac operator. These two ingredients can, in some sense, be combined into the single notion of a  $\text{Spin}^c$ -structure. Such a structure allows us to give another definition of geometric quantisation, which is slightly different from the previous one. We will use this definition in Theorem 6.13 about discrete series representations of semisimple Lie groups.

It is possible to restate Definition 3.20 of Dolbeault-quantisation in terms of  $\text{Spin}^c$ -structures associated to almost complex structures and prequantum line bundles. See for example [79]. This definition is different from the one we give in this section, where we do not use almost complex structures. The difference between these definitions is explained in [62].

#### $\text{Spin}^c$ -structures and Dirac operators

We begin by introducing  $\text{Spin}^c$ -structures on manifolds. More information can be found in [22] or in [53], Appendix D. For  $n \in \mathbb{N}$ ,  $n \geq 2$ , the group  $\text{Spin}(n)$  is by definition the connected double cover of  $\text{SO}(n)$ . It can be constructed explicitly as follows.

The Clifford algebra of a vector space  $V$  with a quadratic form  $q$  is the quotient of the tensor algebra of  $V$  by the two-sided ideal generated by the elements  $v \otimes v - q(v)$ , for  $v \in V$ . See [22, 23, 53] for more information. Let  $C_n$  be the Clifford algebra of  $\mathbb{R}^n$  with the quadratic form  $q(x) = -x_1^2 - \cdots - x_n^2$ . Then  $\text{Spin}(n)$  is the group in  $C_n$  generated by elements of norm one and degree two:

$$\text{Spin}(n) = \langle xy; x, y \in S^{n-1} \subset \mathbb{R}^n \subset C_n \rangle.$$

The group  $\text{Spin}^c(n)$  is defined as

$$\text{Spin}^c(n) := \text{Spin}(n) \times_{\mathbb{Z}_2} \text{U}(1).$$

Here  $\mathbb{Z}_2$  is embedded into  $\text{Spin}(n)$  as the kernel of the covering map  $\lambda : \text{Spin}(n) \rightarrow \text{SO}(n)$ , and into  $\text{U}(1)$  as the subgroup  $\{\pm 1\}$ .

More generally, we have the groups  $\text{Spin}(V)$  and  $\text{Spin}^c(V)$ , for any finite-dimensional vector space  $V$  equipped with a quadratic form. They are defined completely analogously to the groups  $\text{Spin}(n)$  and  $\text{Spin}^c(n)$ , respectively.

**Definition 3.24.** A  $\text{Spin}^c$ -structure on a vector bundle  $E \rightarrow M$  of rank  $r$  is a pair  $(P, \psi)$ , consisting of a right principal  $\text{Spin}^c(n)$ -bundle  $P \rightarrow M$  and a vector bundle isomorphism

$$\psi : P \times_{\text{Spin}^c(r)} \mathbb{R}^r \rightarrow E.$$

Here  $\text{Spin}^c(r)$  acts on  $\mathbb{R}^r$  via the homomorphism  $\text{Spin}^c(r) \rightarrow \text{SO}(r)$  given by  $[a, z] \mapsto \lambda(a)$ , for  $a \in \text{Spin}(r)$  and  $z \in \text{U}(1)$ .

A  $\text{Spin}^c$ -structure on a manifold is a  $\text{Spin}^c$ -structure on its tangent bundle. A manifold equipped with a  $\text{Spin}^c$ -structure is called a  $\text{Spin}^c$ -manifold.

A  $\text{Spin}^c$ -structure on a vector bundle  $E \rightarrow M$  induces a metric and an orientation on  $E$ , obtained from the Euclidean metric and the standard orientation on  $\mathbb{R}^{d_M}$ , via the map  $\psi$ . If  $E$  was already equipped with these structures, then the map  $\psi$  is supposed to preserve them. That is,  $\psi$  is an isometric isomorphism of oriented vector bundles.

If an action of a group  $G$  on  $M$  is given, then an equivariant  $\text{Spin}^c$ -structure on  $M$  is a  $\text{Spin}^c$ -structure  $(P, \psi)$ , where  $G$  acts on  $P$  from the left, and  $\psi$  is assumed to be  $G$ -equivariant.

We will sometimes sloppily use the term  $\text{Spin}^c$ -structure for the principal  $\text{Spin}^c$ -bundle  $P$ .

**Remark 3.25** (Spin-structures). A *Spin-structure* is defined in the same way as a  $\text{Spin}^c$ -structure, with the group  $\text{Spin}^c(r)$  replaced by  $\text{Spin}(r)$  everywhere. A Spin-structure  $P \rightarrow M$  on a vector bundle of rank  $r$  naturally induces a  $\text{Spin}^c$ -structure on this bundle, equal to  $P \times_{\text{Spin}(r)} \text{Spin}^c(r) \rightarrow M$ .

Now suppose  $n \in \mathbb{N}$  is even. We denote the *canonical representation* of  $C_n$  by  $c : C_n \rightarrow \text{End}(\Delta_n)$  (see [53, 22, 23]). The vector space  $\Delta_n$  is naturally isomorphic to  $\mathbb{C}^{2^{n/2}}$ . The restriction to  $\text{Spin}(n)$  of this representation decomposes into two irreducible subrepresentations  $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$  of equal dimension. For  $x \in \mathbb{R}^n \subset C_n$ , we have

$$\begin{aligned} x\Delta_n^+ &:= c(x)\Delta_n^+ \subset \Delta_n^-; \\ x\Delta_n^- &:= c(x)\Delta_n^- \subset \Delta_n^+. \end{aligned} \tag{3.9}$$

The representation  $\Delta_n$  of  $\text{Spin}(n)$  extends to the group  $\text{Spin}^c(n)$  via the formula

$$[a, z] \cdot \delta = z(a \cdot \delta),$$

for  $a \in \text{Spin}(n)$ ,  $z \in \text{U}(1)$  and  $\delta \in \Delta_n$ . The  $\text{Spin}^c$ -Dirac operator acts on sections of the *spinor bundle* associated to the  $\text{Spin}^c$ -structure on  $M$ :

**Definition 3.26.** Let  $(P, \psi)$  be a  $\text{Spin}^c$ -structure on an even-dimensional manifold  $M$ . The *spinor bundle* on  $M$  associated to this  $\text{Spin}^c$ -structure is the vector bundle

$$\mathcal{S} := P \times_{\text{Spin}^c(d_M)} \Delta_{d_M}.$$

The isomorphism  $\Delta_{d_M} \cong \mathbb{C}^{2^{d_M/2}}$  induces a Hermitian metric on  $\mathcal{S}$ . The spinor bundle has a natural decomposition  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ , induced by the decomposition  $\Delta_{d_M} = \Delta_{d_M}^+ \oplus \Delta_{d_M}^-$ .

The action of  $TM$  on  $\mathcal{S}$ , called the *Clifford action* and denoted by  $c_{TM}$ , is defined as follows. Let  $[p, x] \in P \times_{\text{Spin}^c(n)} \mathbb{R}^n \cong TM$  be given. Then for all  $\delta \in \Delta_{d_M}$ , the Clifford action is defined by

$$c_{TM}([p, x])[p, \delta] := [p, x \cdot \delta]. \quad (3.10)$$

Note that by (3.9), the Clifford action interchanges the sub-bundles  $\mathcal{S}^+$  and  $\mathcal{S}^-$ . The induced action of vector fields on sections of the spinor bundle will also be denoted by  $c_{TM}$ .

To define the  $\text{Spin}^c$ -Dirac operator on an even-dimensional manifold  $M$ , we suppose a Hermitian connection  $\nabla$  on the spinor bundle to be given.

**Definition 3.27.** The  $\text{Spin}^c$ -Dirac operator  $\not{D}_M$  on  $M$ , associated to the  $\text{Spin}^c$ -structure  $(P, \psi)$  and the connection  $\nabla$ , is defined by the property that for all orthonormal local frames  $\{e_1, \dots, e_{d_M}\}$  of  $TM$ , we locally have

$$\not{D}_M = \sum_{j=1}^{d_M} c_{TM}(e_j) \nabla_{e_j}.$$

This operator maps sections of  $\mathcal{S}^+$  to sections of  $\mathcal{S}^-$  and vice versa.

The principal symbol  $\sigma_{\not{D}_M}$  of the  $\text{Spin}^c$ -Dirac operator is given by

$$\sigma_{\not{D}_M}(\xi, \delta) = (\xi, ic_{TM}(\xi^*)\delta)$$

Here  $(\xi, \delta) \in \pi_M^* \mathcal{S}$ , with  $\pi_M$  the cotangent bundle projection of  $M$ . The tangent vector  $\xi^* \in TM$  is the one associated to  $\xi$  via the Riemannian metric on  $M$ . The square of this principal symbol is given by scalar multiplication by  $\|\xi\|^2$ , so that  $\sigma_{\not{D}_M}$  is invertible, and the  $\text{Spin}^c$ -Dirac operator is elliptic.

Furthermore, the  $\text{Spin}^c$ -Dirac operator is symmetric with respect to the  $L^2$ -inner product of compactly supported smooth sections of the spinor bundle ([92], [22], page 69). This  $L^2$ -inner product is defined using the volume form on  $M$  associated to the Riemannian metric. Finally, if  $M$  is equipped with a  $G$ -equivariant  $\text{Spin}^c$ -structure, then the spinor bundle has a natural structure of a  $G$ -vector bundle. If the connection on  $\mathcal{S}$  is  $G$ -equivariant, then so is the  $\text{Spin}^c$ -Dirac operator.

## $\text{Spin}^c$ -quantisation

Let  $(M, \omega)$  be a compact symplectic manifold. In the definition of  $\text{Spin}^c$ -quantisation, we use a slightly different notion of prequantisation from the one introduced in Section 3.1. To define Dolbeault-quantisation, we assumed that the cohomology class  $[\omega]$  was integral. For  $\text{Spin}^c$ -quantisation, we assume that the cohomology class

$$[\omega] + \frac{1}{2}c_1(\wedge_{\mathbb{C}}^{0, d_M}(TM, J)) \in H_{\text{dR}}^2(M) \quad (3.11)$$

is integral, for some almost complex structure  $J$  on  $M$ , not necessarily compatible with  $\omega$ . This integrality condition is independent of the choice of  $J$ . Integrality of (3.11) implies in particular that  $[2\omega]$  is an integral cohomology class, so that  $(M, 2\omega)$  is prequantisable.

**Definition 3.28.** A  $\text{Spin}^c$ -prequantisation of  $(M, \omega)$  is a prequantisation  $(L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla)$ , as in Definition 3.4, of the symplectic manifold  $(M, 2\omega)$ . That is, the curvature form of  $\nabla$  is  $4\pi i \omega$  instead of  $2\pi i \omega$ .

Note that if  $L^\omega$  is a normal prequantum line bundle over  $(M, \omega)$ , then  $(L^\omega)^{\otimes 2}$  is a  $\text{Spin}^c$ -prequantum line bundle. We will motivate this definition in Lemma 3.32.

In the case of  $\text{Spin}^c$ -quantisation, the link between the  $\text{Spin}^c$ -structure and the prequantisation is given by the determinant line bundle:

**Definition 3.29.** The *determinant homomorphism*  $\det : \text{Spin}^c(n) \rightarrow \text{U}(1)$  is given by

$$\det[a, z] = z^2,$$

for  $a \in \text{Spin}(n)$  and  $z \in \text{U}(1)$ .

Let  $P \rightarrow M$  be a principal  $\text{Spin}^c(n)$ -bundle. The *determinant line bundle* of  $P$  is the line bundle

$$\det(P) := P \times_{\text{Spin}^c(n)} \mathbb{C} \rightarrow M,$$

where  $\text{Spin}^c(n)$  acts on  $\mathbb{C}$  via the determinant homomorphism.

**Definition 3.30** (Quantisation IV). Let  $(M, \omega)$  be a compact symplectic manifold, and suppose that the cohomology class (3.11) is integral. Then there is a  $\text{Spin}^c$ -prequantisation  $(L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla)$  of  $(M, \omega)$ , and a  $\text{Spin}^c$  structure  $P \rightarrow M$  on  $M$  whose determinant line bundle is (isomorphic to)  $L^{2\omega}$  (see Remark 3.31). Let

$$\not{D}_M^{L^{2\omega}} : \Gamma^\infty(M, \mathcal{S}^+) \rightarrow \Gamma^\infty(M, \mathcal{S}^-)$$

be the  $\text{Spin}^c$ -Dirac operator on the spinor bundle  $\mathcal{S}$ , with respect to any connection on  $\mathcal{S}$ . Its index is the  $\text{Spin}^c$ -quantisation of  $(M, \omega)$ :

$$\boxed{Q_N(M, \omega) := \text{index } \not{D}_M^{L^{2\omega}}.}$$

Note that the principal symbol, and hence the index, of  $\not{D}_M^{L^{2\omega}}$  does not depend on the choice of connection on  $\mathcal{S}$ .

**Remark 3.31.** Integrality of (3.11) implies that a  $\text{Spin}^c$ -structure  $P$  as in Definition 3.30 always exists. Indeed, let  $J$  be any almost complex structure on  $M$ , not necessarily compatible with  $\omega$ . By integrality of (3.11), the line bundle

$$L^{2\omega} \otimes \wedge_{\mathbb{C}}^{0, d_M}(TM, J) \rightarrow M$$

always has a square root  $L_J$ . Then  $P$  may be taken to be the standard  $\text{Spin}^c$ -structure associated to  $L_J$  and  $J$ , as described for example in [27], Proposition D.50.

The specific choice of the  $\text{Spin}^c$ -structure  $P$  is irrelevant in Definition 3.30, as long as its determinant line bundle is  $L^{2\omega}$ .

The link between Definitions 3.4 and 3.28 of prequantisation, and between Definitions 3.20 and 3.30 of geometric quantisation, is the following.

**Lemma 3.32.** *Let  $(M, \omega)$  be a compact symplectic manifold, and let  $L^\omega \rightarrow M$  be a prequantum line bundle. Then  $L^{2\omega} := (L^\omega)^{\otimes 2} \rightarrow M$  is a  $\text{Spin}^c$ -prequantum line bundle. Let  $J$  be an almost complex structure on  $M$ , compatible with  $\omega$ . If the line bundle*

$$\Lambda_{\mathbb{C}}^{0,d_M}(TM, J) \rightarrow M$$

*is trivial, then the Dolbeault-quantisation of  $(M, \omega)$ , with respect to  $L^\omega$ , equals the  $\text{Spin}^c$ -quantisation of  $(M, \omega)$ , with respect to  $L^{2\omega}$ .*

*Sketch of proof.* In the situation of this lemma, the spinor bundle  $\mathcal{S}$  is isomorphic to  $\Lambda^{0,*}T^*M \otimes L^\omega$ , and this isomorphism intertwines the principal symbols of the  $\text{Spin}^c$ - and Dolbeault–Dirac operators (up to a nonzero scalar factor).  $\square$

### 3.5 Equivariant quantisation

So far, we have only defined quantisation in the absence of a group action. These definitions generalise naturally to the equivariant setting. Let  $(M, \omega)$  be a compact symplectic manifold, equipped with a symplectic action by a group  $G$ . Let a ( $\text{Spin}^c$ - or normal) equivariant prequantisation be given.

In the case of Dolbeault-quantisation, let  $J$  be a  $G$ -equivariant almost complex structure on  $M$ , compatible with  $\omega$ . If the action of  $G$  on  $M$  is proper, then such an almost complex structure always exists (see [27], Example D.12 and Corollary B.35). In the case of  $\text{Spin}^c$ -quantisation, the  $\text{Spin}^c$ -structure  $P$  in Definition 3.30 can be given the structure of a  $G$ -equivariant  $\text{Spin}^c$ -structure, by applying the construction in Remark 3.31 to an equivariant almost complex structure on  $M$ . Choose a  $G$ -equivariant connection on the corresponding spinor bundle. It then follows that the virtual vector spaces  $Q_I(M, \omega)$ – $Q_N(M, \omega)$  are invariant under the representation of  $G$  given by (3.4), and therefore carry representations of  $G$ .

If  $G = K$  is a compact Lie group, then these quantisations therefore define elements of the *representation ring* of  $K$ :

**Definition 3.33.** Let  $K$  be a compact Lie group. The *representation ring*  $R(K)$  of  $K$  is the quotient of the free abelian group with one generator for each isomorphism class of finite-dimensional representations of  $K$ , by the equivalence relation  $[V] + [W] \sim [V \oplus W]$ , for all finite-dimensional  $K$ -representations  $V$  and  $W$ . The tensor product of representations induces a commutative product on  $R(K)$ .

In particular, we have

$$Q_{III}(M, \omega) = K\text{-index}(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*) \in R(K); \quad (3.12)$$

$$Q_N(M, \omega) = K\text{-index} \mathcal{D}_M^{L^{2\omega}} \in R(K). \quad (3.13)$$

Here the Dolbeault–Dirac operator  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  and the  $\text{Spin}^c$ -Dirac operator  $\mathcal{D}_M^{L^{2\omega}}$  are understood as operators between the spaces of even- and odd-graded antiholomorphic differential forms with values in  $L^\omega$ , or sections of the spinor bundle, as in Definitions 3.20 and 3.30.

The goal of this thesis is to generalise the ‘quantisation commutes with reduction’ theorem in Section 3.7 to noncompact  $M$  and  $G$ . Definitions (3.12) and (3.13) cannot directly be generalised to this case, for two reasons. The first is that if  $M$  is noncompact, then the kernels of the Dolbeault- and  $\text{Spin}^c$ -Dirac operators need no longer be finite-dimensional. The second reason is that the representation ring has to be defined in terms of finite-dimensional representations, to avoid problems with formal differences of infinite-dimensional vector spaces, and that the finite-dimensional representations of noncompact Lie groups do not include all the interesting ones. Indeed, for noncompact simple groups the only finite-dimensional unitary representations are direct sums of the trivial one. We will use the solution to these problems proposed by Landsman [50], which is to replace the representation ring of a group by the  $K$ -theory of its  $C^*$ -algebra, and the  $K$ -index by the analytic assembly map. This is explained in Chapters 4, 5 and 6.

### 3.6 Quantisation of symplectic reductions

Because we always suppose that the orbit space of a given group action is compact, all symplectic reductions we consider are compact as well. We can therefore quantise these reductions in the usual way, which we describe in this section.

Suppose that  $G$  is a group,  $(M, \omega)$  is a Hamiltonian  $G$ -manifold, with momentum map  $\Phi$ , and that  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  is an equivariant prequantisation. Suppose  $M/G$  is compact. Consider the symplectic reduction  $(M_0, \omega_0)$  of  $(M, \omega)$  at zero. If 0 is a regular value of  $\Phi$ , and  $G$  acts properly and freely on  $\Phi^{-1}(0)$ , then we have the line bundle

$$L^{\omega_0} := (L^\omega|_{\Phi^{-1}(0)})/G \rightarrow M_0. \quad (3.14)$$

If  $p : \Phi^{-1}(0) \rightarrow M_0$  is the quotient map, and  $i : \Phi^{-1}(0) \hookrightarrow M$  is the inclusion, then we have  $p^*L^{\omega_0} \cong i^*L^\omega$ . The  $G$ -invariant Hermitian metric  $(-, -)_{L^\omega}$  induces a metric  $(-, -)_{L^{\omega_0}}$  on  $L^{\omega_0}$ , by

$$(G \cdot l, G \cdot l')_{L^{\omega_0}} := (l, l')_{L^\omega},$$

for all  $m \in \Phi^{-1}(0)$  and  $l, l' \in L_m^\omega$ . Furthermore, there is a unique connection  $\nabla^{M_0}$  on  $L^{\omega_0}$  such that  $p^*\nabla^{M_0} = i^*\nabla$  (see [28], Theorem 3.2). The triple  $(L^{\omega_0}, (-, -)_{L^{\omega_0}}, \nabla^{M_0})$  is a prequantisation of  $(M_0, \omega_0)$ .

To define the Dolbeault-quantisation of the the symplectic reduction  $(M_0, \omega_0)$ , we choose an almost complex structure  $J^{M_0}$  on  $M_0$ , compatible with  $\omega_0$ . We then form the Dolbeault–Dirac operator  $\bar{\partial}_{L^{\omega_0}} + \bar{\partial}_{L^{\omega_0}}^*$  with respect to  $J^{M_0}$ . As in Section 3.3, the Dolbeault-quantisation is the index of this operator:

$$Q_{III}(M_0, \omega_0) = \text{index}(\bar{\partial}_{L^{\omega_0}} + \bar{\partial}_{L^{\omega_0}}^*).$$

For  $\text{Spin}^c$ -quantisation, let  $P \rightarrow M$  be a  $G$ -equivariant  $\text{Spin}^c$ -structure with determinant line bundle  $L^{2\omega}$ . In [64], Paradan shows that  $P$  induces a  $\text{Spin}^c$ -structure  $P_0$  on  $M_0$  whose determinant line bundle is  $L^{2\omega_0}$ . The  $\text{Spin}^c$ -quantisation of  $(M_0, \omega_0)$  is then defined, as in Section 3.4, as the index of the  $\text{Spin}^c$ -Dirac operator on the spinor bundle  $\mathcal{S}_0$  of  $P_0$ , with respect to any connection on  $\mathcal{S}_0$ :

$$Q_N(M_0, \omega_0) = \text{index} \not{D}_{M_0}^{L^{2\omega_0}}.$$

Even if the action of  $G$  on  $\Phi^{-1}(0)$  is not assumed to be free, it is still locally free by Lemma 2.24. If the action of  $G$  on  $\Phi^{-1}(0)$  is proper, then it has compact stabilisers, so that the reduced space  $M_0$  is an orbifold. It is then still possible to define a Dolbeault- or  $\text{Spin}^c$ -Dirac operator on  $M_0$ , and its index is still denoted by  $Q_{III}(M_0, \omega_0)$  or by  $Q_N(M_0, \omega_0)$ , respectively. These indices can be computed via Kawasaki's orbifold index theorem (see [41], or [59], Theorem 3.3).

If 0 is not a regular value of  $\Phi$ , then  $M_0$  is not necessarily an orbifold. In [60], Meinrenken and Sjamaar deal with this situation in the compact setting. Because their methods may not work in the noncompact setting, we will avoid working with such singular spaces by only considering regular values of  $\Phi$ .

Next, let any element  $\xi \in \mathfrak{g}^*$  be given. The  $\text{Spin}^c$ -quantisation  $Q_N(M_\xi, \omega_\xi)$  of the symplectic reduction of  $(M, \omega)$  at  $\xi$  can be defined analogously to the case  $\xi = 0$ .

For Dolbeault-quantisation, suppose  $\xi$  has the property that  $\langle \xi, X \rangle \in 2\pi i\mathbb{Z}$  for all  $X \in \ker \exp$ . Then  $\xi$  lifts to a homomorphism  $e^\xi : G_\xi \rightarrow \text{U}(1)$  (with  $G_\xi$  the stabiliser of  $\xi$  with respect to the coadjoint action). Let  $\mathcal{O}^\xi$  be the coadjoint orbit through  $\xi$ , and consider the line bundle

$$L^{\mathcal{O}^\xi} := G \times_{G_\xi} \mathbb{C}_\xi \rightarrow G/G_\xi \cong \mathcal{O}^\xi,$$

where  $G_\xi$  acts on  $\mathbb{C}_\xi$  via the homomorphism  $e^\xi$ .

By the shifting trick (Remark 2.22), the diagonal action of  $G$  on  $M \times \mathcal{O}^{-\xi}$  is Hamiltonian, and its symplectic reduction at zero is symplectomorphic to  $(M_\xi, \omega_\xi)$ . Consider the exterior product line bundle  $L^\omega \boxtimes L^{\mathcal{O}^{-\xi}}$  over  $M \times \mathcal{O}^{-\xi}$ , with metric and connection induced by those on  $L^\omega$  and some choices of metric and connection on  $L^{\mathcal{O}^{-\xi}}$ . The quantisation  $Q(M_\xi, \omega_\xi)$  is by definition the quantisation of the reduction *at zero* of  $(M \times \mathcal{O}^{-\xi}, \omega \times \omega^{-\xi})$ , prequantised by  $L^\omega \boxtimes L^{\mathcal{O}^{-\xi}}$ , as described above. By homotopy invariance of the index, this quantisation is independent of the choices of the connection and the metric on  $L^{\mathcal{O}^{-\xi}}$ . We will denote the line bundle over  $M_\xi = (M \times \mathcal{O}^{-\xi})_0$  induced by  $L^\omega \boxtimes L^{\mathcal{O}^{-\xi}}$  as in (3.14) by  $L^{\omega_\xi}$ .

### 3.7 Quantisation commutes with reduction: the compact case

In the case of compact Lie groups  $K$ , quantum reduction is easy to define. Indeed, quantum reduction at the trivial representation, denoted by  $R_K^0$  is defined by taking subspaces of  $K$ -invariant vectors:

$$\begin{aligned} R_K^0 : R(K) &\rightarrow \mathbb{Z}; \\ [V] - [W] &\mapsto \dim V^K - \dim W^K, \end{aligned} \tag{3.15}$$

for all finite-dimensional representations  $V$  and  $W$  of  $K$ .

#### Dolbeault-quantisation

With the notion of quantum reduction described above, we have the following ‘quantisation commutes with reduction’ theorem in the case of Dolbeault-quantisation.

**Theorem 3.34** (Dolbeault-quantisation commutes with reduction). *Let  $(M, \omega)$  be a compact Hamiltonian  $K$ -manifold, with momentum map  $\Phi$ . Suppose there is a  $K$ -equivariant prequantisation of  $(M, \omega)$ . If  $0 \in \Phi(M)$ , then*

$$R_K^0(Q_{III}(M, \omega)) = Q_{III}(M_0, \omega_0),$$

with  $Q_{III}$  as in Definition 3.20. If  $0 \notin \Phi(M)$ , then the integer on the left hand side equals zero.

This theorem was proved in various degrees of generality in [38, 59, 60, 63, 79, 84]. The most general proof, without any regularity assumptions on the momentum map or on the group action, is the one given in [60]. If  $Q_{III}$  is replaced by  $Q_I$ , Theorem 3.34 was proved by Guillemin and Sternberg in [28]. After Guillemin and Sternberg published their result, and before Theorem 3.34 was proved in this generality, the latter theorem became known as the *Guillemin–Sternberg conjecture*. An overview is given in [70].

Theorem 3.34 can be symbolically expressed by the ‘quantisation commutes with reduction’-diagram

$$\begin{array}{ccc} (K \curvearrowright M, \omega) & \xrightarrow{Q} & G \curvearrowright Q(M, \omega) \\ \downarrow R_K^0 & & \downarrow R_K^0 \\ (M_0, \omega_0) & \xrightarrow{Q} & Q(M_0, \omega_0) = Q(M, \omega)^K. \end{array} \quad (3.16)$$

Here on the left hand side,  $R_K^0$  denotes symplectic reduction at zero.

Theorem 3.34 admits a generalisation to reduction at other representations than the trivial one. Quantum reduction at an arbitrary irreducible representation  $U$  of  $K$  is defined by taking the multiplicity of  $U$  in a given representation:

$$\begin{aligned} R_K^U : R(K) &\rightarrow \mathbb{Z}; \\ [V] - [W] &\mapsto [V : U] - [W : U]. \end{aligned} \quad (3.17)$$

Here  $[V : U]$  denotes the multiplicity of  $U$  in  $V$ , which by Schur’s lemma equals the dimension of  $\text{Hom}(U, V)^K$ .

To state a ‘quantisation commutes with reduction’ theorem at other irreducible representations than the trivial one, we now apply some representation theory of compact Lie groups to link quantum reduction at a given irreducible representation to symplectic reduction at some element of  $\mathfrak{k}^*$ . Let  $T < K$  be a maximal torus, let  $\mathfrak{t} \subset \mathfrak{k}$  be its Lie algebra, and let  $\mathfrak{t}_+^* \subset \mathfrak{k}^*$  be a choice of positive Weyl chamber. Let  $\Lambda_+ \subset i\mathfrak{t}_+^*$  be the set of dominant integral weights with respect to the positive roots for  $(\mathfrak{k}, \mathfrak{t})$  corresponding to  $\mathfrak{t}_+^*$ . The elements  $\lambda \in \Lambda_+$  are in one-to-one correspondence with the irreducible representations of  $K$ . This correspondence is given by  $\lambda \mapsto V_\lambda$ , where  $V_\lambda$  is the irreducible representation of  $K$  with highest weight  $\lambda$ . We will write  $R_K^\lambda := R_K^{V_\lambda}$  for the reduction map at  $V_\lambda$ , and  $(M_\lambda, \omega_\lambda) := (M_{-i\lambda}, \omega_{-i\lambda})$  for the symplectic reduction of  $(M, \omega)$  at  $-i\lambda \in \mathfrak{k}^* \hookrightarrow \mathfrak{k}^*$ . The embedding  $\mathfrak{k}^* \hookrightarrow \mathfrak{k}^*$  is given by

$$\mathfrak{k}^* \cong (\mathfrak{k}^*)^{\text{Ad}(T)} \subset \mathfrak{k}^*.$$

**Theorem 3.35** (Dolbeault-quantisation commutes with reduction). *Let  $(M, \omega)$  be a compact Hamiltonian  $K$ -manifold, with momentum map  $\Phi$ . Suppose there is a  $K$ -equivariant prequantisation of  $(M, \omega)$ . Then for all  $\lambda \in \Lambda_+ \cap i\Phi(M)$ ,*

$$R_K^\lambda(Q_{III}(M, \omega)) = Q_{III}(M_\lambda, \omega_\lambda),$$



with  $Q_{III}$  as in Definition 3.20. If  $\lambda \notin i\Phi(M)$ , then this integer equals zero.

In other words, we get a complete decomposition

$$Q_{III}(M, \omega) = \bigoplus_{\lambda \in \Lambda_+ \cap i\Phi(M)} Q_{III}(M_\lambda, \omega_\lambda) V^\lambda,$$

of the virtual  $K$ -representation  $Q_{III}(M, \omega)$  into irreducibles.

In the compact case, Theorem 3.35 can be deduced from Theorem 3.34. This deduction is possible because of the shifting trick and the following example.

**Example 3.36** (The Borel–Weil theorem). The Borel–Weil theorem in representation theory is a special case of Theorem 3.35. However, all known proofs of Theorem 3.35 depend on the Borel–Weil theorem to deduce this theorem from Theorem 3.34. Hence the Borel–Weil theorem is not obtained as a corollary to Theorem 3.35, but only serves as an illustration.

To deduce the Borel–Weil theorem from Theorem 3.35, consider Example 2.13 about coadjoint orbits. Let  $\lambda \in \Lambda_+$  be given, and let  $\mathcal{O}^\lambda$  be the coadjoint orbit through  $-i\lambda$ . Note that  $\mathcal{O}^\lambda \cong K/K_\lambda$  as smooth manifolds. There is a complex structure on  $K/K_\lambda$  which gives  $\mathcal{O}^\lambda$  the structure of a Kähler manifold. We have the prequantum line bundle  $L^{\mathcal{O}^\lambda}$  over  $(\mathcal{O}^\lambda, \omega^\lambda)$ , defined as

$$L^{\mathcal{O}^\lambda} = K \times_{K_\lambda} \mathbb{C}_\lambda \rightarrow K/K_\lambda,$$

where  $K_\lambda$  acts on  $\mathbb{C}_\lambda$  via the global weight  $e^\lambda : K_\lambda \rightarrow \text{U}(1)$ . It can be shown that this line bundle is ‘positive enough’, so that by Kodaira’s vanishing theorem, we have  $H^{0,k}(\mathcal{O}^\lambda; L^{\mathcal{O}^\lambda}) = 0$  if  $k > 0$ . Definitions I – III of geometric quantisation therefore coincide in this case, and we see that Theorem 3.35 implies that

$$Q_{III}(\mathcal{O}^\lambda, \omega^\lambda) = V_\lambda.$$

This is a version of the Borel–Weil theorem (see e.g. [85], Theorem 6.3.7). See also [12].

Example 3.36 illustrates the mathematical relevance of Theorem 3.35. This theorem is of mathematical interest because it is a link between symplectic geometry and representation theory. In other words, a link between the *mathematics behind* classical mechanics and the *mathematics behind* quantum mechanics. This mathematical link is a more important reason why the author is interested in Theorem 3.34 than a possible physical link between classical mechanics and quantum mechanics that this theorem may provide.

Using the Borel–Weil theorem, we can show that Theorem 3.35 follows from Theorem 3.34. We will use the fact that

$$Q_{III}(M \times N, \omega \times \nu) = Q_{III}(M, \omega) \otimes Q_{III}(N, \nu) \quad (3.18)$$

for Hamiltonian  $K$ -manifolds  $(M, \omega)$  and  $(N, \nu)$ . This relation follows for example from the Künneth formula for Dolbeault-cohomology.

**Lemma 3.37.** *Theorem 3.34 implies Theorem 3.35.*

*Proof.* Let  $\lambda \in \Lambda_+$  be given. Then using the shifting trick (Remark 2.22), Theorem 3.34 and formula (3.18), we get

$$\begin{aligned} Q_{III}(M_\lambda, \omega_\lambda) &= Q_{III}((M \times \mathcal{O}^{-\lambda})_0, (\omega \times \omega^{-\lambda})_0) \\ &= R_K^0(Q_{III}(M \times \mathcal{O}^{-\lambda}, \omega \times \omega^{-\lambda})) \\ &= (Q_{III}(M, \omega) \otimes Q_{III}(\mathcal{O}^{-\lambda}, \omega^{-\lambda}))^K. \end{aligned}$$

Now by the general form of the Borel–Weil theorem, we have  $Q_{III}(\mathcal{O}^{-\lambda}, \omega^{-\lambda}) = V_\lambda^*$ , so that

$$Q_{III}(M_\lambda, \omega_\lambda) = (Q_{III}(M, \omega) \otimes V_\lambda^*)^K = R_K^\lambda(Q_{III}(M, \omega)).$$

□

See also [60], Corollary 2.11.

## Spin<sup>c</sup>-quantisation

For Spin<sup>c</sup>-quantisation, we have the following result, which is Theorem 1.7 in Paradan’s paper [64].

**Theorem 3.38** (Spin<sup>c</sup>-quantisation commutes with reduction). *Let  $(M, \omega)$  be a compact Hamiltonian  $K$ -manifold, with momentum map  $\Phi$ . Suppose there is a  $K$ -equivariant Spin<sup>c</sup>-prequantisation of  $(M, \omega)$ . Let  $\rho$  be half the sum of the positive roots of  $(\mathfrak{k}, \mathfrak{t})$  with respect to  $\mathfrak{t}_+^*$ .*

*If all stabilisers of the action of  $K$  on  $M$  are abelian, then for all  $\lambda \in \Lambda_+ \cap i\Phi(M)$ ,*

$$R_K^\lambda(Q_N(M, \omega)) = Q_N(M_{\lambda+\rho}, \omega_{\lambda+\rho}),$$

*with  $Q_N$  as in Definition 3.30. If  $\lambda \notin i\Phi(M)$ , then this integer equals zero.*

The condition that the action of  $K$  on  $M$  has abelian stabilisers is related to the fact that there may be several different coadjoint orbits in  $\mathfrak{k}^*$  whose Spin<sup>c</sup>-quantisation equals a given irreducible representation of  $K$ . This ambiguity, which is not present in the case of Dolbeault-quantisation, can be removed by imposing the condition that the action has abelian stabilisers.

## Generalisations

Various generalisations of Theorems 3.34 and 3.38 have been considered. Vergne [83] has found an approach to quantising certain classes of actions by noncompact groups on noncompact manifolds. In [64], Paradan proves a version of the Guillemin–Sternberg conjecture for Hamiltonian actions by compact groups  $K$  on possibly non-compact manifolds, under some assumptions that are satisfied by regular elliptic coadjoint orbits of semisimple groups. He defines the quantisation of such an action as the index of a certain transversally elliptic symbol, which is an element of the generalised character ring  $R^{-\infty}(K)$ . The unpublished work of Duflo and Vargas on restricting discrete series representations of semisimple groups to semisimple subgroups can also be interpreted as a ‘quantisation commutes with reduction’ result for Hamiltonian actions on coadjoint orbits.

Generalising in another direction, Bos [11] defines a notion of Hamiltonian Lie groupoid actions, and proves a Guillemin–Sternberg conjecture for Hamiltonian actions of proper Lie groupoids on bundles of compact Kähler manifolds.

In [50], Landsman proposes a generalisation of Theorem 3.34 to actions by noncompact groups on noncompact manifolds, as long as the orbit space of such an action is compact. This generalisation is formulated in the language of *noncommutative geometry*, as we will explain in Chapters 4, 5 and 6.

The aim of the author’s Ph.D. project was to prove Landsman’s generalisation in as many cases as possible. Part III contains a proof of this generalisation for groups  $G$  that have a discrete normal subgroup  $\Gamma$  such that  $G/\Gamma$  is compact, such as  $G = \mathbb{R}^n$  or  $G$  discrete. In Part IV, we prove a generalisation of Theorem 3.38 for semisimple groups, where  $\lambda$  parametrises the discrete series representations of such a group.

The strategy of the proofs in this thesis is to deduce the noncompact case from the compact case. Thus, Theorems 3.34 and 3.38 are essential ingredients of our proofs, and we do not obtain these theorems as corollaries to our results. The reduction to the compact case is made possible by the ‘naturality of the assembly map’-results that we prove in Part II.

# Chapter 4

## Noncommutative geometry

We will generalise the ‘quantisation commutes with reduction’ results in the compact case, Theorems 3.34 and 3.38, to the noncompact case using tools from noncommutative geometry. These tools are  $K$ -theory and  $K$ -homology of  $C^*$ -algebras. In Chapter 5, we will introduce  $KK$ -theory, which is a powerful tool that generalises both  $K$ -theory and  $K$ -homology. Using  $KK$ -theory, we then define the *analytic assembly map* used in the Baum–Connes conjecture. This map will replace the  $K$ -index in Definitions 3.20 and 3.30 of geometric quantisation.

Further explanations, as well as the proofs we omit, can be found in [10, 17, 18, 23, 52, 87].

### 4.1 $C^*$ -algebras

The central objects of study in noncommutative geometry are  $C^*$ -algebras. Actually, ‘noncommutative topology’ is a more accurate term for the study of  $C^*$ -algebras without further structure. Indeed, the basis of noncommutative geometry is the idea that all information about a locally compact Hausdorff space  $X$  is contained in the algebra  $C_0(X)$  of (complex-valued) continuous functions on  $X$  that ‘vanish at infinity’. These algebras have natural structures of commutative  $C^*$ -algebras, and the central goal in noncommutative geometry is to extend the tools of topology and geometry, such as  $K$ -theory and (co)homology, to noncommutative  $C^*$ -algebras.

#### The basic theory

Let us explain the example of the algebra  $C_0(X)$  in some more detail.

**Example 4.1** (Continuous functions vanishing at infinity). Let  $X$  be a locally compact Hausdorff space. A function  $f$  on  $X$  is said to *vanish at infinity* if for all  $\varepsilon > 0$  there is a compact subset  $C \subset X$  such that for all  $x \in X \setminus C$ , we have  $|f(x)| < \varepsilon$ . The vector space of continuous functions on  $X$  vanishing at infinity is denoted by  $C_0(X)$ . Note that if  $X$  is compact, then all functions on  $X$  vanish at infinity (just take  $C = X$ ).

For  $f, g \in C_0(X)$  and  $x \in X$ , set

$$\begin{aligned}\|f\|_\infty &:= \sup_{y \in X} |f(y)|; \\ f^*(x) &:= \overline{f(x)}; \\ (fg)(x) &= f(x)g(x).\end{aligned}\tag{4.1}$$

Then  $C_0(X)$  is a Banach space in the norm  $\|\cdot\|_\infty$ , and a commutative algebra over  $\mathbb{C}$  with respect to the pointwise product (4.1). Furthermore, we have for all  $f, g \in C_0(X)$ ,

$$\begin{aligned}\|fg\|_\infty &\leq \|f\|_\infty \|g\|_\infty; \\ \|f^*f\|_\infty &= \|f\|_\infty^2.\end{aligned}$$

The structure on  $C_0(X)$  mentioned in Example 4.1, and its properties (apart from commutativity) are the motivation for the following definition.

**Definition 4.2.** A  $C^*$ -algebra is a Banach space  $(A, \|\cdot\|)$ , equipped with an associative bilinear product  $(a, b) \mapsto ab$  and an antilinear map  $a \mapsto a^*$  whose square is the identity, such that for all  $a, b \in A$ , we have

$$\begin{aligned}(ab)^* &= b^*a^*; \\ \|ab\| &\leq \|a\| \|b\|; \\ \|a^*a\| &= \|a\|^2.\end{aligned}$$

A homomorphism of  $C^*$ -algebras is a linear homomorphism of algebras that intertwines star operations. Such homomorphisms are automatically bounded.

It follows from the  $C^*$ -algebra axioms that  $\|a^*\| = \|a\|$  for all  $a$  in a  $C^*$ -algebra.

The following result shows that studying locally compact Hausdorff spaces is equivalent to studying commutative  $C^*$ -algebras. It is proved for example in [18], Theorem 1.4.1.

**Theorem 4.3** (Gelfand–Naimark for commutative  $C^*$ -algebras). *Every commutative  $C^*$ -algebra is isomorphic to the  $C^*$ -algebra of continuous functions that vanish at infinity on a locally compact Hausdorff space. If two commutative  $C^*$ -algebras  $C_0(X)$  and  $C_0(Y)$  are isomorphic, then  $X$  and  $Y$  are homeomorphic.*

A proper continuous map  $f$  between two locally compact Hausdorff spaces  $X$  and  $Y$  induces a homomorphism of  $C^*$ -algebras

$$f^* : C_0(Y) \rightarrow C_0(X),$$

defined by pulling back functions along  $f$ . In this way,  $C_0$  is a contravariant functor from the category of locally compact Hausdorff spaces, with proper continuous maps, to the category of commutative  $C^*$ -algebras. Together with the fact that all homomorphisms between two commutative  $C^*$ -algebras  $C_0(X)$  and  $C_0(Y)$  are defined by pulling back along some proper continuous map, Theorem 4.3 implies that this functor defines an equivalence of categories.

Note that a commutative  $C^*$ -algebra has a unit if and only if the corresponding space is compact. This correspondence will be used in Section 4.2 on  $K$ -theory.

The following example is the standard example of a noncommutative  $C^*$ -algebra.

**Example 4.4.** Let  $\mathcal{H}$  be a Hilbert space, and let  $\mathcal{B}(\mathcal{H})$  be the algebra of bounded operators on  $\mathcal{H}$ . For  $a \in \mathcal{B}(\mathcal{H})$ , let  $\|a\|$  be the operator norm of  $a$ , and let  $a^*$  be its adjoint, defined by

$$(x, ay) = (a^*x, y)$$

for all  $x, y \in \mathcal{H}$ . Then  $\mathcal{B}(\mathcal{H})$ , equipped with these structures, is a  $C^*$ -algebra.

In fact, all  $C^*$ -algebras can be realised as subalgebras of an algebra of bounded operators on a Hilbert space (see [18], Theorem 2.6.1):

**Theorem 4.5** (Gelfand–Naimark for general  $C^*$ -algebras). *Every  $C^*$ -algebra is isomorphic to a norm-closed subalgebra of  $\mathcal{B}(\mathcal{H})$  that in addition is closed under the  $*$ -operation, for some Hilbert space  $\mathcal{H}$ .*

**Example 4.6.** Let  $X$  be a locally compact Hausdorff space. Given a measure on  $X$ , with respect to the Borel  $\sigma$ -algebra of  $X$ , we can form the Hilbert space  $L^2(X)$ . For suitable measures (the counting measure always works), the representation of  $C_0(X)$  in  $L^2(X)$  as multiplication operators yields an embedding of  $C_0(X)$  into  $\mathcal{B}(L^2(X))$ .

## Group $C^*$ -algebras

The two kinds of  $C^*$ -algebras we will use most in this thesis are commutative ones and *group  $C^*$ -algebras*. Let  $G$  be a locally compact Hausdorff topological group, equipped with a left Haar measure  $dg$ . For two functions  $\varphi, \psi \in C_c(G)$ , their *convolution product*  $\varphi * \psi$  is defined by

$$(\varphi * \psi)(g) := \int_G \varphi(g') \psi(g'^{-1}g) dg. \quad (4.2)$$

The function  $\varphi^*$  is defined by

$$\varphi^*(g) := \overline{\varphi(g^{-1})} \Delta(g)^{-1}, \quad (4.3)$$

where  $\Delta$  is the modular function on  $G$  with respect to  $dg$ , defined by  $d(gh) = \Delta(h)dg$  for all  $h \in G$ . We will only consider *unimodular* groups, defined by the property that  $\Delta$  is the constant function 1. In other words, by the property that any left Haar measure is also right invariant (and vice versa).

The full and reduced  $C^*$ -algebras of  $G$  are defined as completions of  $C_c(G)$  in certain norms, with multiplication and  $*$ -operation defined as the continuous extensions of (4.2) and (4.3).

To define these norms, we consider unitary representations  $(\mathcal{H}, \rho)$  of  $G$ . For  $\varphi \in C_c(G)$ , we have the operator

$$\rho(\varphi) := \int_G \varphi(g) \rho(g) dg \in \mathcal{B}(\mathcal{H}).$$

The norm  $\|\cdot\|$  used to define the *full  $C^*$ -algebra*  $C^*(G)$  of  $G$  is

$$\|\varphi\| := \sup_{(\mathcal{H}, \rho) \in \hat{G}} \|\rho(\varphi)\|_{\mathcal{B}(\mathcal{H})}.$$

Here  $\hat{G}$  denotes the *unitary dual* of  $G$ , i.e. the set of all irreducible unitary representations of  $G$ . This supremum is finite, because  $\|\rho(\varphi)\|_{\mathcal{B}(\mathcal{H})} \leq \|\varphi\|_{L^1(G)}$  for all  $\varphi \in C_c(G)$  and all unitary representations  $(\mathcal{H}, \rho)$  of  $G$ .

The *reduced  $C^*$ -algebra*  $C_r^*(G)$  of  $G$  is the completion of  $C_c(G)$  in the norm  $\|\cdot\|_r$ , given by

$$\|\varphi\|_r := \|\lambda^G(\varphi)\|_{\mathcal{B}(L^2(G))}.$$

Here  $\lambda^G : G \rightarrow \mathcal{U}(L^2(G))$  is the left regular representation

$$(\lambda^G(g)\varphi)(g') = \varphi(g^{-1}g').$$

Note that  $\lambda^G(\varphi)\psi = \varphi * \psi$  for all  $\varphi \in C_c(G)$  and  $\psi \in L^2(G)$ .

The convolution product on  $C^*(G)$  and  $C_r^*(G)$  is commutative if and only if  $G$  is commutative. Hence, by Theorem 4.3, for abelian groups  $G$ , there are locally compact Hausdorff spaces  $X$  and  $Y$  such that

$$\begin{aligned} C^*(G) &\cong C_0(X); \\ C_r^*(G) &\cong C_0(Y). \end{aligned} \tag{4.4}$$

It turns out that both  $X$  and  $Y$  may be taken to be the unitary dual  $\hat{G}$  of  $G$ . The isomorphisms (4.4) are given by the Fourier transform.

So for abelian groups  $G$ , we have  $C^*(G) = C_r^*(G)$ . This equality also holds if  $G$  is compact (but not necessarily abelian). Indeed, by the Peter–Weyl theorem ([46], Theorem IV.4.20) every irreducible representation of a compact group  $G$  occurs in the left regular representation of  $G$  in  $L^2(G)$ . In general, a group is called *amenable* if its full and reduced  $C^*$ -algebras are equal.

The  $C^*$ -algebra of a compact Lie group can be described explicitly as follows. We will use this description in the proof of Proposition 4.29. Let  $K$  be a compact Lie group, and consider the direct sum

$$\bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi), \tag{4.5}$$

where, as before,  $\hat{K}$  is the set of irreducible (unitary) representations  $(V_\pi, \pi)$  of  $K$ , and this direct sum by definition consists of the sequences  $(a_\pi)_{\pi \in \hat{K}}$  such that  $a_\pi \in \mathcal{B}(V_\pi)$  for all  $\pi$ , and

$$\lim_{\pi \rightarrow \infty} \|a_\pi\|_{\mathcal{B}(V_\pi)} = 0.$$

(That is, for all  $\varepsilon > 0$ , there is a finite set  $X \subset \hat{K}$  such that  $\|a_\pi\|_{\mathcal{B}(V_\pi)} < \varepsilon$  for all  $\pi$  outside  $X$ .) Equipped with the norm

$$\|(a_\pi)_{\pi \in \hat{K}}\| := \sup_{\pi \in \hat{K}} \|a_\pi\|_{\mathcal{B}(V_\pi)}$$

and the natural  $*$ -operation, (4.5) becomes a  $C^*$ -algebra.

**Proposition 4.7.** *There is an isomorphism of  $C^*$ -algebras*

$$C^*(K) \cong \bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi). \tag{4.6}$$

*Sketch of proof.* Consider the Hilbert space

$$\hat{L}^2(K) := \left\{ a = (a_\pi)_{\pi \in \hat{K}} \in \prod_{\pi \in \hat{K}} \mathcal{B}(V_\pi); (a, a) := \sum_{\pi \in \hat{K}} \text{tr}(a_\pi^* a_\pi) < \infty \right\}.$$

It follows from the Peter–Weyl theorem (see e.g. [46], Theorem 4.20) that the Plancherel transform  $P : L^2(K) \rightarrow \hat{L}^2(K)$ , given by

$$(Pf)_\pi = \sqrt{\dim V_\pi} \pi(f)$$

for  $f \in L^2(K)$  and  $\pi \in \hat{K}$ , is a unitary isomorphism. Consider the map  $\varphi : C^*(K) \rightarrow \mathcal{B}(\hat{L}^2(K))$  that on  $C(K)$  is given by

$$\varphi(f) = P\pi(f)P^{-1},$$

and extended continuously to all of  $C^*(K)$ . This map can be shown to be an isomorphism of  $C^*$ -algebras onto its image, which is the right hand side of (4.6), acting on  $\hat{L}^2(K)$  by left multiplication.  $\square$

## Additional concepts

We conclude this section with some definitions that we will use occasionally.

**Definition 4.8.** A  $C^*$ -algebra is said to be  $\sigma$ -unital if it has a countable approximate unit. That is, there is a sequence  $(e_j)_{j=1}^\infty$  in  $A$ , such that for all  $a \in A$ , the sequences  $(e_j a)_{j=1}^\infty$  and  $(a e_j)_{j=1}^\infty$  converge to  $a$ .

**Example 4.9.** Full and reduced group  $C^*$ -algebras are  $\sigma$ -unital; a sequence in  $C_c(G)$  that converges to the distribution  $\delta_e$  is an approximate identity.

A commutative  $C^*$ -algebra  $C_0(X)$  is  $\sigma$ -unital if  $X$  is  $\sigma$ -compact. If  $(C_j)_{j=1}^\infty$  is an increasing collection of compact subsets of  $X$  such that  $\bigcup_{j=1}^\infty C_j = X$ , then an approximate identity can be constructed as a sequence of functions in  $C_c(X)$  such that the  $j$ th function equals 1 on  $C_j$ .

**Definition 4.10.** Let  $A$  be a  $C^*$ -algebra. By Theorem 4.5, it can be embedded into the algebra of bounded operators on some Hilbert space  $\mathcal{H}$ . The *multiplier algebra* of  $A$  is the algebra

$$M(A) := \{T \in \mathcal{B}(\mathcal{H}); TA \subset A \text{ and } AT \subset A\}.$$

**Example 4.11.** Let  $X$  be a locally compact Hausdorff space, and consider the  $C^*$ -algebra  $C_0(X)$  as an algebra of operators on  $L^2(X)$ , for some measure on  $X$ . Then  $M(C_0(X)) = C_b(X)$ , the  $C^*$ -algebra of continuous bounded functions on  $X$ . Being a unital  $C^*$ -algebra, the algebra  $C_b(X)$  equals  $C(\beta X)$  for some compact Hausdorff space  $X$ , called the *Stone–Čech compactification* of  $X$ .

The following property of multiplier algebras will play a role in the definition of the homomorphism  $V_N$  (see page 104).

**Lemma 4.12.** Any homomorphism of  $C^*$ -algebras  $A \rightarrow B$  extends to a homomorphism  $M(A) \rightarrow M(B)$ .

See [87], Proposition 2.2.16.

In particular, any representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  of a  $C^*$ -algebra  $A$  in a Hilbert space  $\mathcal{H}$  extends to a representation

$$\pi : M(A) \rightarrow M(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}).$$



**Definition 4.13.** A *positive element* of a  $C^*$ -algebra  $A$  is an element  $a \in A$  for which there exists an element  $b \in A$  such that  $a = b^*b$ .

**Example 4.14.** If  $\mathcal{H}$  is a Hilbert space, then a positive element of  $\mathcal{B}(\mathcal{H})$  is an element  $a$  such that

$$(x, ax) \geq 0$$

for all  $x \in \mathcal{H}$ .

The tensor product of two  $C^*$ -algebras  $A$  and  $B$  can be formed in several ways, that is, with respect to several different norms on the algebraic tensor product  $A \otimes B$ . See [87], Appendix T for more information.

**Definition 4.15.** The *minimal tensor product*  $A \otimes_{\min} B$  is the completion of the algebraic tensor product  $A \otimes B$  as a subalgebra of  $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , if  $A$  and  $B$  are realised as algebras of bounded operators on two Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. The resulting norm on  $A \otimes_{\min} B$  is denoted by  $\|\cdot\|_{\min}$ .

**Definition 4.16.** The *maximal tensor product*  $A \otimes_{\max} B$  is the completion of the algebraic tensor product  $A \otimes B$  in the norm

$$\left\| \sum_k a_k \otimes b_k \right\|_{\max} := \sup \left\| \sum_k \pi_A(a_k) \pi_B(b_k) \right\|_{\mathcal{B}(\mathcal{H})}, \quad (4.7)$$

for  $a_k \in A$  and  $b_k \in B$ , where the supremum is taken over all commuting representations  $\pi_A : A \rightarrow \mathcal{B}(\mathcal{H})$  and  $\pi_B : B \rightarrow \mathcal{B}(\mathcal{H})$  of  $A$  and  $B$  on the same Hilbert space  $\mathcal{H}$ .

The supremum in (4.7) actually turns out to be a maximum.

For any norm  $\|\cdot\|$  on  $A \otimes B$  with the property that the completion in this norm is a  $C^*$ -algebra, one has

$$\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max},$$

which explains the names of these norms. A  $C^*$ -algebra  $A$  is called *nuclear* if for all other  $C^*$ -algebras  $B$ , the minimal and maximal norms on  $A \otimes B$  coincide. Then there is only one way to form the tensor product of  $A$  with any other given  $C^*$ -algebra (if this tensor product is required to be a  $C^*$ -algebra).

**Example 4.17.** Commutative  $C^*$ -algebras are nuclear. In particular, one has

$$C_0(X_1) \otimes C_0(X_2) \cong C_0(X_1 \times X_2) \quad (4.8)$$

for all locally compact Hausdorff spaces  $X_1$  and  $X_2$ .

**Example 4.18.** For all locally compact Hausdorff groups  $G_1$  and  $G_2$ , one has

$$\begin{aligned} C^*(G_1) \otimes_{\max} C^*(G_2) &\cong C^*(G_1 \times G_2); \\ C_r^*(G_1) \otimes_{\min} C_r^*(G_2) &\cong C_r^*(G_1 \times G_2). \end{aligned} \quad (4.9)$$

## 4.2 $K$ -theory

One of the nicest results in noncommutative topology is the generalisation of Atiyah–Hirzebruch topological  $K$ -theory for locally compact Hausdorff spaces, i.e. commutative  $C^*$ -algebras, to arbitrary  $C^*$ -algebras. We begin with the definition of topological  $K$ -theory, and then we rephrase this definition in a  $C^*$ -algebraic way. This allows us to generalise the definition to arbitrary  $C^*$ -algebras.

### Topological $K$ -theory

We first consider a *compact* Hausdorff space  $X$ .

**Definition 4.19.** The (*topological*)  $K$ -theory of  $X$  is the abelian group  $K^0(X)$  whose generators are isomorphism classes  $[E]$  of (complex) vector bundles over  $X$ , subject to the relation

$$[E] + [F] = [E \oplus F]$$

for all vector bundles  $E$  and  $F$  over  $X$ .

A continuous map  $f : X \rightarrow Y$  between compact Hausdorff spaces induces a map  $f^* : K^0(Y) \rightarrow K^0(X)$ , defined via the pullback of vector bundles along  $f$ . This turns  $K^0$  into a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

More information about topological  $K$ -theory can be found in [3]. Note that a general element of  $K^0(X)$  is a formal difference  $[E] - [F]$  of isomorphism classes of vector bundles.

Vector bundles over locally compact, but not compact spaces are not as well-behaved as those over compact spaces. Therefore, the  $K$ -theory of a general locally compact Hausdorff space  $X$  is not defined directly as in Definition 4.19, but via the one-point compactification  $X^+$  of  $X$ .

Let  $X^+ = X \cup \{\infty\}$  be the one-point compactification of  $X$ . Let

$$i : \{\infty\} \hookrightarrow X^+$$

be the inclusion map of the point at infinity. Consider the functorially induced map

$$i^* : K^0(X^+) \rightarrow K^0(\{\infty\}).$$

Note that vector bundles over the one-point space  $\{\infty\}$  are just finite-dimensional vector spaces, whose isomorphism classes are characterised by their dimensions. Therefore  $K^0(\{\infty\}) \cong \mathbb{Z}$ .

**Definition 4.20.** The  $K$ -theory of the locally compact Hausdorff space  $X$  is the kernel of the map  $i^*$ . It is denoted by  $K^0(X)$ .

As a consequence of this definition, the only maps between locally compact Hausdorff spaces that induce maps on  $K$ -theory are the ones that extend to continuous maps between one-point compactifications. These are the *proper* continuous maps. Hence topological  $K$ -theory is a contravariant functor from the category of locally compact Hausdorff spaces, with proper continuous maps, to the category of abelian groups. (See also the remark below Theorem 4.3.)

### $K$ -theory of unital $C^*$ -algebras

Let us rephrase the definition of  $K^0(X)$  in terms of the  $C^*$ -algebra  $C_0(X)$ . First suppose that  $X$  is compact, so that  $C_0(X)$  equals the algebra  $C(X)$  of all continuous functions on  $X$ .

If  $E \rightarrow X$  is a vector bundle, then the space  $\Gamma(E)$  of its continuous sections has the natural structure of a  $C(X)$ -module, given by pointwise multiplication. Two such  $C(X)$ -modules  $\Gamma(E)$  and  $\Gamma(F)$  are isomorphic if and only if  $E \cong F$  as vector bundles. Note that there is a natural isomorphism of  $C(X)$ -modules  $\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$ . Furthermore, for any vector bundle  $E \rightarrow X$ , there is a vector bundle  $F \rightarrow X$  such that  $E \oplus F$  is trivial, say isomorphic to  $X \times \mathbb{R}^n$  (see [3], Corollary 1.4.14). This implies that

$$\Gamma(E) \oplus \Gamma(F) \cong \Gamma(E \oplus F) = \Gamma(X \times \mathbb{R}^n) \cong C(X)^n.$$

More generally, a module  $\mathfrak{M}$  over a  $C^*$ -algebra (or ring)  $A$  is called *finitely generated and projective* if there exists an  $A$ -module  $\mathfrak{N}$  such that  $\mathfrak{M} \oplus \mathfrak{N}$  is a finitely generated free  $A$ -module, i.e. of the form  $A^n$  for some  $n \in \mathbb{N}$ . It turns out that *any* finitely generated projective  $C(X)$ -module is isomorphic to the module  $\Gamma(E)$ , for some vector bundle  $E \rightarrow X$ . Hence Definition 4.19 of  $K$ -theory for compact spaces can be restated as follows:

**Proposition 4.21** (Serre–Swan). *The  $K$ -theory of the compact Hausdorff space  $X$  is the abelian group whose generators are isomorphism classes  $[\mathfrak{M}]$  of finitely generated projective  $C(X)$ -modules, subject to the relation*

$$[\mathfrak{M}] + [\mathfrak{N}] = [\mathfrak{M} \oplus \mathfrak{N}]$$

*for all finitely generated projective modules  $\mathfrak{M}$  and  $\mathfrak{N}$  over  $C(X)$ .*

The definition of topological  $K$ -theory provided by Proposition 4.21 can be generalised to arbitrary  $C^*$ -algebras with a unit.

**Definition 4.22.** Let  $A$  be a  $C^*$ -algebra with a unit. The  $K$ -theory of  $A$  is the group in Proposition 4.21, with  $C(X)$  replaced by  $A$ . This abelian group is denoted by  $K_0(A)$ .

A unital homomorphism  $f : A \rightarrow B$  of unital  $C^*$ -algebras induces a map  $f_* : K_0(A) \rightarrow K_0(B)$ . This map is defined by  $[\mathfrak{M}] \mapsto [\mathfrak{M} \otimes_f B]$ , for finitely generated projective (right)  $A$ -modules  $\mathfrak{M}$ . The tensor product  $\mathfrak{M} \otimes_f B$  is the algebraic tensor product  $\mathfrak{M} \otimes B$  over  $\mathbb{C}$ , with the equivalence relation

$$(m \cdot a) \otimes b \sim m \otimes (f(a)b),$$

for all  $m \in \mathfrak{M}$ ,  $a \in A$  and  $b \in B$ , divided out. This makes the  $K$ -theory of unital  $C^*$ -algebras a covariant functor. By Lemma 4.23 below, this functoriality generalises the functoriality of topological  $K$ -theory for compact spaces.

Note that this time we use a subscript 0 instead of a superscript, because we are dealing with a covariant functor on  $C^*$ -algebras, rather than a contravariant functor on topological spaces.

**Lemma 4.23.** *Let  $X$  and  $Y$  be compact Hausdorff spaces, let  $f : X \rightarrow Y$  be a continuous map, and let  $E \rightarrow Y$  be a vector bundle. Consider the homomorphism of  $C^*$ -algebras  $f^* : C(Y) \rightarrow C(X)$  defined by pulling back functions along  $f$ . There is an isomorphism*

$$\Gamma(X, f^*E) \cong \Gamma(Y, E) \otimes_{f^*} C(X).$$

See [23], Proposition 2.12.

### $K$ -theory of general $C^*$ -algebras

The extension of Definition 4.22 to possibly non-unital  $C^*$ -algebras is analogous to the extension of Definition 4.19 to Definition 4.20. Indeed, if  $X$  is a locally compact Hausdorff space, then

$$C_0(X) \oplus \mathbb{C} \cong C(X^+).$$

The isomorphism is given by  $(f, z) \mapsto \tilde{f} + z$ , where  $\tilde{f} \in C(X^+)$  is given by

$$\begin{aligned}\tilde{f}(x) &= f(x) \quad \text{for all } x \in X; \\ \tilde{f}(\infty) &= 0.\end{aligned}$$

The multiplication, star operation and the norm on  $C_0(X) \oplus \mathbb{C}$  are defined by

$$\begin{aligned}(f + z)(g + w) &:= fg + zg + wg + zw; \\ (f + z)^* &:= f^* + \bar{z}; \\ \|f + z\| &:= \max_{y \in X^+} |f(y) + z| \\ &= \sup_{x \in X} |f(x) + z| \\ &= \|f + z\|_{\mathcal{B}(C_0(X))},\end{aligned}$$

for  $f, g \in C_0(X)$  and  $z, w \in \mathbb{C}$ . The resulting  $C^*$ -algebra is called the *unitisation* of  $C_0(X)$ .

The inclusion map  $i : \{\infty\} \hookrightarrow X^+$  induces the map

$$i^* : C_0(X) \oplus \mathbb{C} \cong C_0(X^+) \rightarrow \mathbb{C} \quad (4.10)$$

given by the natural projection onto the term  $\mathbb{C}$ . Then we have

**Proposition 4.24.** *The topological  $K$ -theory of  $X$  is the kernel of the map*

$$i_* := (i^*)_* : K_0(C(X^+)) \rightarrow K_0(\mathbb{C}) \cong \mathbb{Z}$$

induced by (4.10).

For a general  $C^*$ -algebra, we proceed as follows.

**Definition 4.25.** Let  $(A, \|\cdot\|_A)$  be a  $C^*$ -algebra. Its *unitisation*  $A^+$  is defined as the algebra  $A^+ := A \oplus \mathbb{C}$ , with multiplication, star operation and norm given by

$$\begin{aligned}(a + z)(b + w) &:= ab + zb + wa + zw; \\ (a + z)^* &:= a^* + \bar{z}; \\ \|a + z\|_{A^+} &:= \|a + z\|_{\mathcal{B}(A)},\end{aligned}$$

for  $a, b \in A$  and  $z, w \in \mathbb{C}$ . Here  $\|a + z\|_{\mathcal{B}(A)}$  is the norm of  $a + z$  as a bounded operator on the Banach space  $A$ , given by left multiplication.

For a  $C^*$ -algebra  $A$ , consider the map

$$\begin{aligned} i^* : A^+ &\rightarrow \mathbb{C}, \\ a + z &\mapsto z. \end{aligned}$$

We denote the induced map on  $K$ -theory by

$$i_* := (i^*)_* : K_0(A^+) \rightarrow K_0(\mathbb{C}) \cong \mathbb{Z}.$$

**Definition 4.26.** The  $K$ -theory of  $A$  is the kernel of the map  $i_*$ . It is denoted by  $K_0(A)$ .

Hence for all locally compact Hausdorff spaces, we have  $K^0(X) = K_0(C_0(X))$ .

For unital  $A$ , Definition 4.26 reduces to Definition 4.22. Note that for any  $C^*$ -algebra  $A$ , every finitely generated projective  $A$ -module can be extended to an finitely generated projective  $A^+$ -module, which is in the kernel of the map  $i_*$ . Such modules therefore define classes in  $K_0(A)$ , as in the unital case, although they usually do not exhaust the whole group  $K_0(A)$ .

**Remark 4.27** ( $K$ -theory via projections). The  $K$ -theory of a unital  $C^*$ -algebra  $A$  is often defined using projections in the ‘infinite matrix algebra’

$$M_\infty(A) := \varinjlim M_n(A),$$

i.e. elements  $p$  such that  $p^2 = p = p^*$ . These correspond to projective  $A$ -modules via  $p \mapsto p(A^n)$ , for  $p$  a projection in  $M_n(A)$ . The functoriality of  $K$ -theory is then induced by

$$f(p)_{ij} = f(p_{ij}) \quad \in B,$$

if  $f : A \rightarrow B$  is a homomorphism of  $C^*$ -algebras and  $p \in M_\infty(A)$  is a projection.

By the way, in this picture another reason why  $K$ -theory for non-unital  $C^*$ -algebras has to be defined separately becomes apparent. Indeed, if  $X$  is a connected, locally compact but not compact Hausdorff space, then there are no nonzero projections in  $M_\infty(C_0(X))$ , because the trace of such a projection is a constant function on  $X$ .

**Remark 4.28** (Higher  $K$ -groups). For any integer  $n$ , and any  $C^*$ -algebra  $A$ , one has the  $K$ -theory group  $K_n(A) := K_0(A \otimes C_0(\mathbb{R}^n))$ . *Bott periodicity* is the statement that  $K_{n+2}(A) \cong K_n(A)$  for all such  $n$  and  $A$  (naturally in  $A$ ). Therefore, it is enough to consider the  $K$ -theory groups  $K_0(A)$  and  $K_1(A)$ . In this thesis, we will only use  $K_0(A)$ . This is eventually related to the fact that we consider symplectic, and hence even-dimensional manifolds.

## The $K$ -theory of the $C^*$ -algebra of a compact group

The only  $C^*$ -algebras whose  $K$ -theory we will use in this thesis are (full or reduced) group  $C^*$ -algebras (see Section 4.1). For compact groups  $K$ , this  $K$ -theory group<sup>1</sup> is isomorphic to

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<sup>1</sup>This is one of the few occasions where we use the capital letter  $K$  to denote both a compact group and a  $K$ -theory functor. We hope this is not too confusing.

the abelian group underlying the representation ring  $R(K)$ . Indeed, let  $(V_\pi, \pi)$  be a finite-dimensional representation of  $K$ . Then  $V_\pi$  has the structure of a projective  $C^*(K)$ -module, given by

$$f \cdot v := \rho(f)v = \int_K f(k)\pi(k)v dk. \quad (4.11)$$

Here  $f \in C(K)$ ,  $v \in V_\pi$ ,  $dk$  is a Haar measure on  $K$ , and this  $C(K)$ -module structure on  $V$  extends continuously to a  $C^*(K)$ -module structure.

**Proposition 4.29.** *This procedure induces an isomorphism of abelian groups*

$$R(K) \cong K_0(C^*(K)). \quad (4.12)$$

*Proof.* The proof of this proposition is based on Proposition 4.7, which states that

$$C^*(K) \cong \bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi). \quad (4.13)$$

Let a sequence  $(X_n)_{n=1}^\infty$  of finite subsets of  $\hat{K}$  be given, such that  $X_n \subset X_{n+1}$  for all  $n$ , and that  $\bigcup_{n=1}^\infty X_n = \hat{K}$ . Then it follows from the definition of inductive limits of Banach algebras ([10], Section 3.3) that

$$\bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi) = \varinjlim \bigoplus_{\pi \in X_n} \mathcal{B}(V_\pi).$$

We conclude that, by continuity of  $K$ -theory with respect to inductive limits (see [10], 5.2.4 or [52], Theorem 6.3.2),

$$\begin{aligned} K_0(C^*(K)) &\cong \varinjlim K_0\left(\bigoplus_{\pi \in X_n} \mathcal{B}(V_\pi)\right) \\ &= \varinjlim \bigoplus_{\pi \in X_n} K_0(\mathcal{B}(V_\pi)) \\ &= \bigoplus_{\pi \in \hat{K}} K_0(\mathcal{B}(V_\pi)) \\ &= \bigoplus_{\pi \in \hat{K}} \mathbb{Z} \cdot [V_\pi] \\ &= R(K). \end{aligned}$$

In the second line from the bottom,  $V_\pi$  is first viewed as a  $\mathcal{B}(V_\pi)$ -module, and then as an irreducible representation of  $K$ . The fact that the resulting isomorphism  $K_0(C^*(K)) \cong R(K)$  is given by the procedure described above Proposition 4.29, follows from the explicit form of the isomorphism (4.13), as given in the proof of Proposition 4.7.  $\square$

Recall that for compact groups, the full and reduced  $C^*$ -algebras coincide.

Proposition 4.29 is crucial to the motivation of the definition of quantisation we will use (Definition 6.1). This quantisation takes values in the  $K$ -theory group of the (full or reduced)  $C^*$ -algebra of the group in question. By Proposition 4.29, this corresponds to an element of the representation ring in the case of compact groups.

## 4.3 $K$ -homology

As we said at the end of the previous section, the quantisation procedure we use takes values in the  $K$ -theory of the group that acts on the symplectic manifold that is to be quantised. In the case of compact groups and manifolds, geometric quantisation was defined as the equivariant index of a Dirac operator. In the noncompact case, the  $K$ -theory element that is the quantisation of a symplectic action will be the ‘generalised equivariant index’ of an ‘abstract elliptic operator’ defined by the same Dirac operator. To be more precise, the ‘abstract elliptic operators’ on a  $G$ -space  $X$  will be the elements of the  $K$ -homology group  $K_0^G(X)$  defined in this section. The ‘generalised equivariant index’ of such an element is its image under the *analytic assembly map*, which is defined in Section 5.2.

### The definition of $K$ -homology

We begin with the abstract definition of the  $K$ -homology group  $K_0^G(X)$ . We will later state a theorem that (some) first order elliptic differential operators on a smooth manifold define elements of the associated  $K$ -homology group. The Dirac operators that we use to define quantisation are examples of such elliptic operators.

Let  $X$  be a locally compact Hausdorff space. Let  $G$  be a locally compact Hausdorff topological group acting properly on  $X$ .

**Definition 4.30.** 1. An *equivariant  $K$ -homology cycle*, or *equivariant abstract elliptic operator* over  $X$  is a triple  $(\mathcal{H}, F, \pi)$ , where

- $\mathcal{H}$  is a  $\mathbb{Z}_2$ -graded Hilbert space carrying a graded unitary representation of  $G$  (such as the space  $L^2(E)$ , for some  $\mathbb{Z}_2$ -graded Hermitian  $G$ -vector bundle  $E \rightarrow X$ , with respect to some measure on  $X$ );
- $F$  is a bounded operator on  $\mathcal{H}$  which is odd with respect to the grading (such as an odd zeroth order pseudo-differential operator on  $E$ , when  $X$  and  $E$  are smooth);
- $\pi$  is a graded representation of  $C_0(X)$  in  $\mathcal{H}$  (such as the pointwise multiplication operator of  $C_0(X)$  on  $L^2(E)$ ).

The triple  $(\mathcal{H}, F, \pi)$  is supposed to satisfy the assumptions that for all  $g \in G$  and  $f \in C_0(X)$ , we have

$$\pi(g \cdot f) = g\pi(f)g^{-1}, \quad (4.14)$$

and the operators  $[F, \pi(f)]$ ,  $\pi(f)(F^2 - 1)$  and  $\pi(f)[g, F]$  are compact.

2. Two  $K$ -homology cycles  $(\mathcal{H}, F, \pi)$  and  $(\mathcal{H}', F', \pi')$  are said to be *unitarily equivalent* if there is a unitary isomorphism  $\mathcal{H} \cong \mathcal{H}'$  that intertwines the representations of  $G$  and of  $C_0(X)$  on  $\mathcal{H}$  and  $\mathcal{H}'$ , as well as the operators  $F$  and  $F'$ .
3. Two  $K$ -homology cycles  $(\mathcal{H}, F, \pi)$  and  $(\mathcal{H}, F', \pi)$  are called *operator homotopic* if there is a continuous path  $(F_t)_{t \in [0,1]}$  in  $\mathcal{B}(\mathcal{H})$  such that  $(\mathcal{H}, F_t, \pi)$  is a  $K$ -homology cycle for all  $t$ , and  $F_0 = F$ ,  $F_1 = F'$ .
4. The *equivariant  $K$ -homology* of  $X$  is the abelian group  $K_0^G(X)$  with one generator for every unitary equivalence class of equivariant  $K$ -homology cycles over  $X$ , with the relations

- $[\mathcal{H}, F, \pi] = [\mathcal{H}, F', \pi]$  if  $(\mathcal{H}, F, \pi)$  and  $(\mathcal{H}, F', \pi)$  are operator homotopic;
- $[\mathcal{H} \oplus \mathcal{H}', F \oplus F', \pi \oplus \pi'] = [\mathcal{H}, F, \pi] + [\mathcal{H}', F', \pi']$ .

$K$ -homology is a covariant functor on the category of locally compact Hausdorff proper  $G$ -spaces with equivariant continuous proper maps: such a map  $f : X \rightarrow Y$  induces a map

$$f_* : K_0^G(X) \rightarrow K_0^G(Y),$$

given by

$$[\mathcal{H}, F, \pi] \mapsto [\mathcal{H}, F, \pi \circ f^*].$$

As with  $K$ -theory, we also have an odd version  $K_1^G$  of  $K$ -homology. We will not use this odd part, however.

## Functional calculus

An operator in a  $K$ -homology cycle is supposed to be bounded, and can be thought of as an abstract zeroth order pseudo-differential operator. We will mainly consider  $K$ -homology classes defined by Dirac operators, which are first-order differential operators. These do not define bounded operators on the space of  $L^2$ -sections of the spinor bundle, and hence do not directly define a  $K$ -homology class. A way to associate a  $K$ -homology class to an unbounded operator is to use *functional calculus* to turn this unbounded operator into a bounded one.

An (*unbounded*) operator on a Hilbert space  $\mathcal{H}$  is a linear map

$$D : \text{dom } D \rightarrow \mathcal{H},$$

where  $\text{dom } D \subset \mathcal{H}$  is a dense subspace. The operator  $D$  is *symmetric* if for all  $x, y \in \text{dom } D$ ,

$$(Dx, y)_{\mathcal{H}} = (x, Dy)_{\mathcal{H}}.$$

The *adjoint* of  $D$  is the operator  $D^*$  with domain

$$\text{dom } D^* := \{x \in \mathcal{H}; \text{ the linear function } y \mapsto (x, Dy)_{\mathcal{H}} \text{ on } \text{dom } D \text{ is bounded}\},$$

and defined by  $(D^*x, y)_{\mathcal{H}} = (x, Dy)_{\mathcal{H}}$  for all  $x \in \text{dom } D^*$  and  $y \in \text{dom } D$ . The operator  $D$  is called *self-adjoint* if  $\text{dom } D^* = \text{dom } D$ , and  $D^* = D$  on this common domain. Functional calculus is defined for self-adjoint operators  $D$ . For any bounded measurable function  $f$  on the spectrum of  $D$ , it allows us to define a bounded operator  $f(D)$  in a suitable way. See for example [68], page 261 for the definition of this operator.

A symmetric operator that is not self-adjoint sometimes has a self-adjoint *closure*. An operator  $D$  on  $\mathcal{H}$  is *closable* if the closure of its graph in  $\mathcal{H} \times \mathcal{H}$  is again the graph of an operator  $\overline{D}$  on  $\mathcal{H}$ . This operator  $\overline{D}$  is then called the *closure* of  $D$ . The domain of  $\overline{D}$  is the completion of  $\text{dom } D$  in the norm  $\|\cdot\|_D$ , which is defined by

$$\|x\|_D^2 := \|x\|_{\mathcal{H}}^2 + \|Dx\|_{\mathcal{H}}^2, \quad (4.15)$$

for all  $x \in \text{dom } D$ .

If the closure of  $D$  is self-adjoint, then we call  $D$  *essentially self-adjoint*, and we can apply the functional calculus to  $\overline{D}$ . We will usually write  $f(D)$  instead of  $f(\overline{D})$  if  $D$  is essentially self-adjoint.

The following result about functional calculus of unbounded operators follows directly from the definition as given for example in [68], page 261.



**Lemma 4.31.** *Let  $\mathcal{H}$  be a Hilbert space, and let  $D : \text{dom} D \rightarrow \mathcal{H}$  be a self-adjoint operator. Let  $\mathcal{H}'$  be another Hilbert space, and let  $T : \mathcal{H} \rightarrow \mathcal{H}'$  be a unitary isomorphism. Let  $f$  be a measurable function on  $\mathbb{R}$ . Then*

$$T f(D) T^{-1} = f(T D T^{-1}).$$

### $K$ -homology classes of first order elliptic differential operators

To define a  $K$ -homology class associated to an essentially self-adjoint elliptic differential operator  $D$ , we will use the operator  $b(D)$ , where  $b$  is a *normalising function*:

**Definition 4.32.** A *normalising function* is a smooth function  $b : \mathbb{R} \rightarrow [-1, 1]$  with the properties that

- $b$  is *odd*;
- $b(x) > 0$  for all  $x > 0$ ;
- $\lim_{x \rightarrow \pm\infty} b(x) = \pm 1$ .

The most common normalising function used in the context of  $K$ -homology is  $b(x) = \frac{x}{\sqrt{1+x^2}}$ . This function has the technical disadvantage that the operator  $b(D)$  need not be *properly supported*, which is required to apply the analytic assembly map to the associated  $K$ -homology class. More on this in Section 5.2.

We are now prepared to define the  $K$ -homology class associated to a symmetric first order elliptic differential operator. Let  $M$  be a smooth manifold, on which a locally compact Hausdorff topological group  $G$  acts properly. Let  $E = E^+ \oplus E^- \rightarrow M$  be a  $\mathbb{Z}_2$ -graded  $G$ -vector bundle, equipped with a  $G$ -invariant Hermitian metric, and let

$$D : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E)$$

be a  $G$ -equivariant first order elliptic differential operator that maps sections of  $E^+$  to sections of  $E^-$  and vice versa. Suppose that  $M$  is equipped with a  $G$ -invariant measure, and consider the unbounded operator  $D : \Gamma_c^\infty(E) \rightarrow L^2(E)$  on  $L^2(E)$ . Suppose it is symmetric. Then it is closable and essentially self-adjoint ([34], Lemma 10.2.1 and Corollary 10.2.6). We can therefore form the bounded operator  $b(D)$  on  $L^2(E)$ , where  $b$  is a normalising function. Finally, let

$$\pi^M : C_0(M) \rightarrow \mathcal{B}(L^2(E))$$

be the representation defined by pointwise multiplication of sections by functions.

The manifold  $M$  is said to be *complete for  $D$*  if there is a *proper* function  $f \in C^\infty(M)$  such that  $[D, f] \in \mathcal{B}(L^2(E))$ .

**Theorem 4.33.** *If  $M$  is complete for  $D$ , then  $(L^2(E), b(D), \pi^M)$  is an equivariant  $K$ -homology cycle over  $X$ . Its  $K$ -homology class is independent of the choice of  $b$ .*

*Proof.* See [34], Theorem 10.6.5 for the non-equivariant case. The equivariant case then follows from Lemma 4.31. □

We denote this  $K$ -homology class by  $[D]$ .

**Remark 4.34.** Two elliptic operators  $D_0$  and  $D_1$  on the same vector bundle, as in Theorem 4.33, define the same class in  $K$ -homology if they have the same principal symbol. Indeed, in that case, the operator  $D_t := tD_1 + (1-t)D_0$  satisfies the assumptions of Theorem 4.33 for all  $t \in [0, 1]$ , and we obtain a homotopy between  $[D_0]$  and  $[D_1]$ .

**Remark 4.35.** In the situation of Theorem 4.33, it is possible to define a  $K$ -homology class  $[D]$  associated to  $D$  in an appropriate way, even if  $M$  is not complete for  $D$  (see [34], Proposition 10.8.2). However, this class does not have the explicit form  $[D] = [L^2(E), b(D), \pi^M]$  that it has if  $M$  is complete for  $D$ . We use this form in the proof of Corollary 8.11, and therefore we always assume that this completeness condition is satisfied.

Our main application of Theorem 4.33 is the following.

**Corollary 4.36.** *Let  $M$  be an even-dimensional manifold, acted on by a locally compact Hausdorff group  $G$ . Suppose  $M$  has a  $G$ -equivariant  $\text{Spin}^c$ -structure, and let  $\mathcal{S}$  be the associated spinor bundle. The  $\text{Spin}^c$ -structure on  $M$  induces a  $G$ -invariant Riemannian metric on  $M$ . This metric induces a  $G$ -invariant density on  $M$ , which we use to define  $L^2$ -sections of  $\mathcal{S}$ .*

*Let  $\mathcal{D}_M$  be the  $\text{Spin}^c$ -Dirac operator on  $M$ , defined using any  $G$ -equivariant Hermitian connection on  $\mathcal{S}$ . If  $M$  is complete as a metric space, then  $\mathcal{D}_M$  satisfies the conditions of Theorem 4.33, and hence defines a class  $[\mathcal{D}_M] \in K_0^G(X)$ .*

*Proof.* The Dirac operator is elliptic, symmetric, and odd with respect to the grading on  $\mathcal{S}$  (see e.g. [20], Lemma 5.5). By the description of the geodesic distance on  $M$  in terms of Dirac operator as given in [17], Chapter VI.1, we see that completeness of  $M$  as a metric space implies that  $M$  is complete for  $\mathcal{D}_M$ .  $\square$

A similar result holds for the Dolbeault–Dirac operator on an almost complex Riemannian manifold.

**Remark 4.37.** The principal symbol of the Dirac operator  $\mathcal{D}_M$  does not depend on the choice of the connection on  $\mathcal{S}$ . Hence the class  $[\mathcal{D}_M]$  is independent of this choice, by Remark 4.34.

We have seen that a Dirac operator defines an abstract elliptic operator in the sense of  $K$ -homology. We will define quantisation as the ‘generalised equivariant index’ of this abstract elliptic operator. This generalised equivariant index is the analytic assembly map, which we will define in Section 5.2. It is defined in terms of  $KK$ -theory, which is a powerful tool that generalises both  $K$ -homology and  $K$ -theory.

# Chapter 5

## *KK*-theory and the assembly map

Kasparov's *KK*-theory is a bivariant functor that assigns an abelian group  $KK_0(A, B)$  to two  $C^*$ -algebras  $A$  and  $B$ . If  $G$  is a group acting on  $A$  and  $B$  in a reasonable way, then we also have the equivariant *KK*-theory group  $KK_0^G(A, B)$  of  $A$  and  $B$ . As in the case of  $K$ -theory and  $K$ -homology, *KK*-theory has an even and an odd part, and we will only use the even part.

There are three useful features of *KK*-theory that we will use in this thesis.

1. *KK*-theory generalises both  $K$ -theory and  $K$ -homology, in the sense that

$$KK_0^G(C_0(X), \mathbb{C}) = K_0^G(X) \quad (5.1)$$

for all locally compact Hausdorff proper  $G$ -spaces  $X$ , and

$$KK_0(\mathbb{C}, B) \cong K_0(B) \quad (5.2)$$

for all  $\sigma$ -unital  $C^*$ -algebras  $B$  (such as group  $C^*$ -algebras).

2. Using *KK*-theory, we can define the *analytic assembly map*

$$\mu_X^G : K_0^G(X) \rightarrow K_0(C_{(r)}^*(G))$$

(for a locally compact Hausdorff space  $X$  equipped with a proper action by a locally compact Hausdorff group  $G$ , such that  $X/G$  is compact) as a map

$$\boxed{\mu_X^G : K_0^G(X) \rightarrow KK_0(\mathbb{C}, C_{(r)}^*(G)),}$$

via the isomorphism (5.2). Here  $C_{(r)}^*(G)$  denotes either the reduced or the full  $C^*$ -algebra of  $G$ .

3. There is a product on *KK*-theory, the most general form of which is a map

$$KK_0^{G_1}(A_1, B_1 \otimes C) \times KK_0^{G_2}(C \otimes A_2, B_2) \xrightarrow{\times_C} KK_0^{G_1 \times G_2}(A_1 \otimes A_2, B_1 \otimes B_2), \quad (5.3)$$

for groups  $G_1$  and  $G_2$ ,  $G_1$ - $C^*$ -algebras  $A_1$  and  $B_1$ , a  $C^*$ -algebra  $C$ , and  $G_2$ - $C^*$ -algebras  $A_2$  and  $B_2$ . Here one can use any tensor product of  $C^*$ -algebras. This general form is defined via the special case where  $B_1 = A_2 = \mathbb{C}$ .

The product (5.3), called the *Kasparov product*, is functorial in many respects, and associative in a suitable sense. We will mainly use this product in the proof of Theorem 9.1.

The construction of  $KK$ -theory was motivated by index theory, and in particular by a desire to find generalisations and more elegant proofs of the Atiyah–Singer index theorem. One result of this desire was the construction of the analytic assembly map, which is our main application of  $KK$ -theory, and is treated in Section 5.2. In Section 5.3, we introduce Baaj and Julg’s unbounded picture of  $KK$ -theory, and describe the analytic assembly map in this setting. This description will be used in the proof of Theorem 9.3 about multiplicativity of the assembly map with respect to the Kasparov product.

## 5.1 The definition of $KK$ -theory

Because the definition of  $KK$ -theory is quite involved, we will try to be as brief as possible about this definition. This section may therefore seem like a big pile of unmotivated definitions on first reading, and we suggest that readers who are not yet familiar with  $KK$ -theory skim through this section, and later return to look at the details when they are needed. We will almost only be concerned with the special cases (5.1) and (5.2), with  $B$  the  $C^*$ -algebra of a group. We will therefore rarely use the machinery of this chapter in its full generality.

More information on  $KK$ -theory can be found in [10, 33], and in Kasparov’s original papers [39, 40].

In this section, all  $C^*$ -algebras are supposed to be *separable*. A commutative  $C^*$ -algebra  $C_0(X)$  is separable if  $X$  is metrisable. Because we usually work with smooth manifolds, this condition is not an important restriction.

### Hilbert $C^*$ -modules

The basic objects in the definition of  $KK$ -theory are the *adjointable operators* on *Hilbert modules* over  $C^*$ -algebras.

**Definition 5.1.** Let  $A$  be a  $C^*$ -algebra. A (right) *Hilbert  $A$ -module* is a (complex) vector space  $\mathcal{E}$ , equipped with the structure of a right  $A$ -module, and with an ‘ $A$ -valued inner product’

$$(-, -)_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \rightarrow A,$$

which is additive in both entries, and has the following properties:

- for all  $e, f \in \mathcal{E}$  and  $a \in A$ , we have  $(e, fa)_{\mathcal{E}} = (e, f)_{\mathcal{E}} a$ ;
- for all  $e, f \in \mathcal{E}$ , we have  $(e, f)_{\mathcal{E}} = (f, e)_{\mathcal{E}}^*$ ;
- for all  $e \in \mathcal{E}$ , the element  $(e, e)_{\mathcal{E}} \in A$  is positive;
- $\mathcal{E}$  is complete in the norm  $\|\cdot\|_{\mathcal{E}}$ , defined by  $\|e\|_{\mathcal{E}}^2 = \|(e, e)_{\mathcal{E}}\|_A$ , for  $e \in \mathcal{E}$ .

A homomorphism of Hilbert  $A$ -modules is a homomorphism of  $A$ -modules that preserves the  $A$ -valued inner products. An isomorphism is a bijective homomorphism.

The tensor product  $\mathcal{E}_1 \otimes \mathcal{E}_2$  of a Hilbert  $A_1$ -module  $\mathcal{E}_1$  and a Hilbert  $A_2$ -module  $\mathcal{E}_2$  is the algebraic tensor product of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  as complex vector spaces, completed in the  $A_1 \otimes A_2$ -valued inner product

$$(e_1 \otimes e_2, e'_1 \otimes e'_2)_{\mathcal{E}_1 \otimes \mathcal{E}_2} := (e_1 \otimes e'_1)_{\mathcal{E}_1} \otimes (e_2 \otimes e'_2)_{\mathcal{E}_2}.$$

Here  $e_j, e'_j \in \mathcal{E}_j$ , and one has to specify which tensor product is used to form  $A_1 \otimes A_2$ .

Note that a Hilbert  $\mathbb{C}$ -module is nothing more than a Hilbert space. The motivating example for the definition of Hilbert modules over  $C^*$ -algebras is the following.

**Example 5.2.** Let  $X$  be a locally compact Hausdorff space, and let  $E$  be a vector bundle over  $X$ , with a Hermitian structure  $(-, -)_E$ . Let  $\Gamma_0(E)$  be the space of continuous sections  $s$  of  $E$  such that the function  $x \mapsto (s(x), s(x))_E$  vanishes at infinity. Then  $\Gamma_0(E)$  is a Hilbert  $C_0(X)$ -module, whose module structure is given by pointwise multiplication, and with the  $C_0(X)$ -valued inner product

$$(s, t)_{\Gamma_0(E)}(x) := (s(x), t(x))_E,$$

for all  $s, t \in \Gamma_0(E)$  and  $x \in X$ .

The algebras of bounded and compact operators on a Hilbert space have the following generalisations to Hilbert  $C^*$ -modules.

**Definition 5.3.** Let  $A$  be a  $C^*$ -algebra, and let  $\mathcal{E}$  be a Hilbert  $A$ -module. The algebra  $\mathcal{B}(\mathcal{E})$  of *adjointable operators* on  $\mathcal{E}$  consists of the  $\mathbb{C}$ -linear  $A$ -module homomorphisms  $T : \mathcal{E} \rightarrow \mathcal{E}$  for which there is another such homomorphism  $T^*$  that satisfies

$$(Te, f)_{\mathcal{E}} = (e, T^*f)_{\mathcal{E}}$$

for all  $e, f \in \mathcal{E}$ .

All adjointable operators are bounded with respect to the norm  $\|\cdot\|_{\mathcal{E}}$ , and  $\mathcal{B}(\mathcal{E})$  is a  $C^*$ -algebra in the operator norm ([10], Proposition 13.2.2).

**Definition 5.4.** The subalgebra  $\mathcal{F}(\mathcal{E}) \subset \mathcal{B}(\mathcal{E})$  of *finite rank operators* on  $\mathcal{E}$  is by definition algebraically generated by operators of the form

$$\theta_{e_1, e_2} : e_3 \mapsto e_1(e_2, e_3)_{\mathcal{E}},$$

for  $e_1, e_2 \in \mathcal{E}$ . The  $C^*$ -algebra  $\mathcal{K}(\mathcal{E})$  of *compact operators* on  $\mathcal{E}$  is by definition the closure of  $\mathcal{F}(\mathcal{E})$  in  $\mathcal{B}(\mathcal{E})$ .

## Kasparov bimodules

The basic building blocks of  $KK$ -theory are the Kasparov bimodules.

**Definition 5.5.** Let  $A$  and  $B$  be  $C^*$ -algebras. A *Kasparov  $(A, B)$ -bimodule* is a triple  $(\mathcal{E}, F, \pi)$ , where

- $\mathcal{E}$  is a countably generated Hilbert  $B$ -module;
- $\pi : A \rightarrow \mathcal{B}(\mathcal{E})$  is a homomorphism of  $C^*$ -algebras;
- $F \in \mathcal{B}(\mathcal{E})$  is an adjointable operator such that for all  $a \in A$ , the operators  $[F, \pi(a)]$ ,  $(F - F^*)\pi(a)$  and  $(F^2 - 1_{\mathcal{E}})\pi(a)$  are *compact*.

One says that  $F$  ‘almost commutes with  $\pi$ ’, is ‘almost self-adjoint’, and ‘almost Fredholm’.<sup>1</sup>

To define equivariant  $KK$ -theory, we will use  $\mathbb{Z}_2$ -graded Kasparov bimodules, equipped with suitable actions by a group  $G$ .

**Definition 5.6.** A  $\mathbb{Z}_2$ -graded Hilbert module over a  $C^*$ -algebra  $A$  is a Hilbert  $A$ -module  $\mathcal{E}$  with a decomposition  $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ , such that  $ae \in \mathcal{E}^k$  for all  $a \in A$  and  $e \in \mathcal{E}^k$ .

A  $\mathbb{Z}_2$ -grading on a Hilbert module  $\mathcal{E}$  naturally induces  $\mathbb{Z}_2$ -gradings on the  $\mathbb{C}^*$ -algebras  $\mathcal{B}(\mathcal{E})$  and  $\mathcal{K}(\mathcal{E})$ .

For the remainder of this section, let  $G$  be a locally compact Hausdorff group that is second countable, i.e. whose topology has a countable basis.

**Definition 5.7.** A  $G$ - $C^*$ -algebra is a  $C^*$ -algebra equipped with a continuous (left)  $G$ -action. If  $A$  is a  $G$ - $C^*$ -algebra, then a  $G$ -Hilbert  $A$ -module is a Hilbert  $A$ -module equipped with a continuous (left) action of  $G$  by bounded, invertible  $\mathbb{C}$ -linear operators, such that

- for all  $e, e' \in \mathcal{E}$  and  $g \in G$ , one has  $(g \cdot e, g \cdot e')_{\mathcal{E}} = g \cdot (e, e')_{\mathcal{E}}$ ;
- for all  $g \in G$ ,  $e \in \mathcal{E}$  and  $a \in A$ , we have  $g \cdot (ea) = (g \cdot e)(g \cdot a)$ .

The only  $G$ - $C^*$ -algebras we will use in this thesis are of the form  $C_0(X)$ , where  $X$  is a  $G$ -space.

A  $\mathbb{Z}_2$ -graded  $G$ -Hilbert module is just what the name means, with the requirement that the  $G$ -action respects the grading.

**Definition 5.8.** Let  $A$  and  $B$  be  $G$ - $C^*$ -algebras. A  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodule is Kasparov  $(A, B)$ -bimodule  $(\mathcal{E}, F, \pi)$ , with the additional properties that

- $\mathcal{E}$  is a  $\mathbb{Z}_2$ -graded  $G$ -Hilbert  $B$ -module;
- $\pi : A \rightarrow \mathcal{B}(\mathcal{E})$  is a  $G$ -equivariant homomorphism of  $C^*$ -algebras that respect the gradings, where  $G$  acts on  $\mathcal{B}(\mathcal{E})$  via conjugation;
- $F \in \mathcal{B}(\mathcal{E})$  reverses the grading on  $\mathcal{E}$  and has the properties that the map  $g \mapsto gFg^{-1}$  from  $G$  to  $\mathcal{B}(\mathcal{E})$  is norm-continuous, and is ‘almost equivariant’, in the sense that for all  $g \in G$  and  $a \in A$ , the operator  $(gFg^{-1} - F)\pi(a)$  is compact.

## The definition

We continue using the notation of Definition 5.8. The equivariant  $KK$ -theory of  $A$  and  $B$  is the set of  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodules, modulo *unitary equivalence* and *homotopy*.

**Definition 5.9.** Two  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodules  $(\mathcal{E}_0, F_0, \pi_0)$  and  $(\mathcal{E}_1, F_1, \pi_1)$  are said to be

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<sup>1</sup>A bounded operator  $F$  on a Hilbert space  $\mathcal{H}$  is called *Fredholm* if there is a bounded operator  $F'$  on  $\mathcal{H}$  such that the operators  $FF' - 1_{\mathcal{H}}$  and  $F'F - 1_{\mathcal{H}}$  are compact. Fredholm operators have finite-dimensional kernels and cokernels, which makes them the central objects of study in index theory.

- *unitarily equivalent* if there is a  $G$ -equivariant isomorphism of Hilbert  $B$ -modules  $\mathcal{E}_0 \cong \mathcal{E}_1$  that respects the gradings, and intertwines  $F_0$  and  $F_1$ , and  $\pi_0(a)$  and  $\pi_1(a)$ , for all  $a \in A$ ;
- *homotopic* if there is a  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, C([0, 1], B))$ -bimodule  $(\mathcal{E}, F, \pi)$ , with the following property. For  $j = 0, 1$ , let  $\text{ev}_j : C([0, 1], B) \rightarrow B$  be the evaluation map at  $j$ . Then, for  $j = 0, 1$ , the  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodule

$$(\mathcal{E} \otimes_{\text{ev}_j} B, F \otimes 1_B, \pi \otimes 1_B)$$

has to be unitarily equivalent to  $(\mathcal{E}_j, F_j, \pi_j)$ . Here  $\mathcal{E} \otimes_{\text{ev}_j} B$  is the tensor product  $\mathcal{E} \otimes B$  over  $\mathbb{C}$ , modulo the equivalence relation  $e\varphi \otimes b \sim e \otimes \text{ev}_j(\varphi)b$ , for all  $e \in \mathcal{E}$ ,  $\varphi \in C([0, 1], B)$  and  $b \in B$ .

**Remark 5.10.** A special case of homotopy of  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodules is *operator homotopy*. This is the fact that two  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodules  $(\mathcal{E}, F, \pi)$  and  $(\mathcal{E}, F', \pi)$  are homotopic if there is a norm-continuous map  $t \mapsto F_t$  from  $[0, 1]$  to  $\mathcal{B}(\mathcal{E})$  such that for all  $t$ ,  $(\mathcal{E}, F_t, \pi)$  is a  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodule, and  $F_0 = F$  and  $F_1 = F'$ .

If  $A$  is separable and  $B$  is  $\sigma$ -unital, then the combined equivalence relation unitary equivalence & operator homotopy is the same as the homotopy equivalence relation ([10], Theorem 18.5.3).

**Definition 5.11.** The *equivariant  $KK$ -theory* of  $A$  and  $B$  is the abelian group  $KK_0^G(A, B)$  of  $\mathbb{Z}_2$ -graded equivariant Kasparov  $(A, B)$ -bimodules modulo homotopy, with addition induced by the direct sum. The inverse is given by

$$-(\mathcal{E}^0 \oplus \mathcal{E}^1, F, \pi) = (\mathcal{E}^1 \oplus \mathcal{E}^0, -F, \pi).$$

Functoriality of  $KK$ -theory is defined as follows. If  $f : A_1 \rightarrow A_2$  is an equivariant homomorphism of  $\mathbb{Z}_2$ -graded  $G$ - $C^*$ -algebras, then for all  $B$ , we have the map  $f^* : KK_0^G(A_2, B) \rightarrow KK_0^G(A_1, B)$ , given by

$$f^*[\mathcal{E}, F, \pi] = [\mathcal{E}, F, \pi \circ f].$$

If, on the other hand,  $\psi : B_1 \rightarrow B_2$  is such a homomorphism, then for all  $A$ , the map  $\psi_* : KK_0^G(A, B_1) \rightarrow KK_0^G(A, B_2)$  is given by

$$\psi_*[\mathcal{E}, F, \pi] = [\mathcal{E} \otimes_{\psi} B, F \otimes 1_B, \pi \otimes 1_B]$$

Thus,  $KK_0^G$  is a contravariant functor in the first variable, and a covariant functor in the second one.

If the group  $G$  is trivial, we omit it from the notation and write  $KK_0(A, B) := KK_0^{\{e\}}(A, B)$ .

## Properties of $KK$

It follows directly from the definitions, and Remark 5.10, that if  $X$  is a locally compact Hausdorff space on which  $G$  acts properly, then

$$KK_0^G(C_0(X), \mathbb{C}) = K_0^G(X),$$

the equivariant  $K$ -homology of  $X$ . In general, the equivariant  $K$ -homology of a  $G$ - $C^*$ -algebra  $A$  is defined as

$$K_G^0(A) := KK_0^G(A, \mathbb{C}).$$

On the other hand, we have

**Theorem 5.12.** *If  $B$  is a  $\sigma$ -unital  $C^*$ -algebra, then*

$$KK_0(\mathbb{C}, B) \cong K_0(B). \quad (5.4)$$

See [10], Proposition 17.5.5 and Theorem 18.5.3.

For unital  $B$ , the isomorphism (5.4) is given by the map defined as follows. First note that for any Hilbert  $B$ -module  $\mathcal{E}$ , there is only one possible  $C^*$ -algebra homomorphism  $\mathbb{C} \rightarrow \mathcal{B}(\mathcal{E})$ . Therefore, a Kasparov  $(\mathbb{C}, B)$ -module may be denoted by  $(\mathcal{E}, F)$ . The isomorphism is given by

$$[\mathcal{E}, F] \mapsto [\ker \tilde{F}^+] - [\ker \tilde{F}^-] \in K_0(B),$$

where  $\tilde{F} = \begin{pmatrix} 0 & \tilde{F}^- \\ \tilde{F}^+ & 0 \end{pmatrix}$  is an operator on  $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ , homotopic to  $F$ , such that  $\ker \tilde{F}^+$  and  $\ker \tilde{F}^-$  are finitely generated projective  $B$ -modules. Existence of such an operator  $\tilde{F}$  can be deduced from Mingo's generalisation of Kuiper's theorem. See [87], Corollary 16.7, Theorem 16.8 and Theorem 17.3.11.

The final, and possibly most important feature of  $KK$ -theory is the existence of the Kasparov product (5.3). We will not define this product here, since its definition is even more technical than the rest of this section. Thorough discussions of this product can be found in [10], Chapter 18, in [33], and in Kasparov's own papers [39, 40].

We will only use some properties of the Kasparov product, the most important of which is its simpler form in the *unbounded picture* of  $KK$ -theory, as described in Section 5.3, in the special case where  $C = \mathbb{C}$ .

## 5.2 The analytic assembly map

The analytic assembly map is a generalisation of the equivariant index of elliptic differential operators on compact manifolds, acted on by compact groups. It is the key ingredient of the *Baum–Connes conjecture*.

### The definition of the assembly map

Let  $X$  be a locally compact Hausdorff space, on which a second countable, locally compact Hausdorff group  $G$  acts properly. Suppose that the orbit space  $X/G$  is compact, i.e. that the action of  $G$  on  $X$  is *cocompact*. The *(analytic) assembly map* is the map

$$\mu_X^G : K_0^G(X) \rightarrow K_0(C^*(G)),$$

or more precisely,

$$\mu_X^G : K_0^G(X) \rightarrow KK_0(\mathbb{C}, C^*(G)),$$



given by

$$\mu_X^G[\mathcal{H}, F, \pi] = [\mathcal{E}, F_{\mathcal{E}}],$$

with  $\mathcal{E}$  and  $F_{\mathcal{E}}$  defined as follows.

Consider the subspace

$$\mathcal{H}_c := \pi(C_c(X))\mathcal{H} \subset \mathcal{H}.$$

Define the  $C_c(G)$ -valued inner product  $(-, -)_{\mathcal{E}}$  on  $\mathcal{H}_c$  by setting

$$(\xi, \eta)_{\mathcal{E}}(g) = (\xi, g \cdot \eta)_{\mathcal{H}},$$

for all  $\xi, \eta \in \mathcal{H}_c$  and  $g \in G$ . Let  $\|\cdot\|_{\mathcal{E}}$  be the associated norm on  $\mathcal{H}_c$ , as in Definition 5.1, with  $A = C^*(G)$ . Then  $\mathcal{E}$  is the completion of  $\mathcal{H}_c$  in this norm. The (right)  $C^*(G)$ -module structure on  $\mathcal{E}$  is given by

$$\xi \cdot f = \int_G f(g)g \cdot \xi dg,$$

for  $\xi \in \mathcal{H}_c$ ,  $f \in C_c(G)$ , and by continuous extension. The  $C^*(G)$ -valued inner product on  $\mathcal{E}$  is the continuous extension of  $(-, -)_{\mathcal{E}}$ .

To define the operator  $F_{\mathcal{E}}$  on  $\mathcal{E}$  induced by  $F$ , we need  $F$  to have the following property.

**Definition 5.13.** The operator  $F$  is called *properly supported* if for every  $f \in C_c(X)$  there is an  $h \in C_c(X)$  such that

$$\pi(h)F\pi(f) = F\pi(f).$$

If  $\mathcal{H}$  is a space of sections of a vector bundle over  $X$ , and  $\pi$  is defined by pointwise multiplication, then  $F$  is properly supported if it is ‘local’, in the sense that it maps compactly supported sections to compactly supported sections. It is always possible to choose  $F$  so that it is properly supported, without changing the corresponding  $K$ -homology class (see also the remark after Definition (3.6) in [8]):

**Lemma 5.14.** For all  $K$ -homology classes  $[\mathcal{H}, F, \pi] \in K_G^0(X)$ , there is an operator  $\tilde{F} \in \mathcal{B}(\mathcal{H})$  which is properly supported and  $G$ -equivariant, such that  $(\mathcal{H}, \tilde{F}, \pi)$  is an equivariant  $K$ -homology cycle over  $X$ , and that  $[\mathcal{H}, F, \pi] = [\mathcal{H}, \tilde{F}, \pi]$ .

*Sketch of proof.* Let  $f \in C_c(X)$  be a function such that for all  $x \in X$ ,

$$\int_G f^2(gx)dg = 1$$

(see Lemma 7.8). Set

$$\tilde{F} = A_f^G(F) := \int_G g\pi(f)F\pi(f)g^{-1}dg.$$

Then  $\tilde{F}$  is a bounded, properly supported,  $G$ -equivariant operator on  $\mathcal{H}$  (see Lemma 7.11, with  $N$  replaced by  $G$ ). It can be shown that  $F$  and  $\tilde{F}$  are homotopic, so that the claim follows.  $\square$

**Remark 5.15.** The only  $K$ -homology classes we will use are those associated to equivariant elliptic differential operators (see Theorem 4.33). The operators in these classes are equivariant by Lemma 4.31, and they are even properly supported for suitable choices of normalising functions (see Proposition 8.3). We will therefore never have to use Lemma 5.14. We have included it so that we can define the analytic assembly map on general  $K$ -homology cycles.

If  $F$  is properly supported, then it maps  $\mathcal{H}_c$  into itself. We will show (Lemma 7.7) that if  $F$  is also equivariant, the restriction of  $F$  to  $\mathcal{H}_c$  is adjointable with respect to the inner product  $(-, -)_{\mathcal{E}}$ , so that it induces an adjointable operator on  $\mathcal{E}$ . This is the operator  $F_{\mathcal{E}}$ .

**Remark 5.16.** There is also a version of the assembly map that takes values in the  $K$ -theory of the *reduced*  $C^*$ -algebra of  $G$ . It is defined in the same way as above, with  $C^*(G)$  replaced by  $C_r^*(G)$  everywhere. We will use the same notation  $\mu_X^G$  for these two versions, since this will usually not cause too much confusion.

The assembly maps for the full and reduced group  $C^*$ -algebras are related as follows. The identity map on  $C_c(G)$  is bounded as a map

$$(C_c(G), \|\cdot\|_{C^*(G)}) \rightarrow (C_c(G), \|\cdot\|_{C_r^*(G)}).$$

Hence it extends to a continuous map  $C^*(G) \rightarrow C_r^*(G)$ , which in turn induces a map on  $K$ -theory

$$\lambda_G : K_0(C^*(G)) \rightarrow K_0(C_r^*(G)).$$

It follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc} K_0^G(X) & \xrightarrow{\mu_X^G} & K_0(C^*(G)) \\ & \searrow \mu_X^G & \downarrow \lambda_G \\ & & K_0(C_r^*(G)). \end{array}$$

The assumption that  $X/G$  is compact is needed to prove that the assembly map is well-defined. If this condition is not satisfied, then it is still possible to define the assembly map on the *representable*  $K$ -homology of  $X$ :

$$RK_0^G(X) := \varinjlim_{A \subset X} K_0^G(A),$$

where  $A$  runs over the  $G$ -invariant subsets  $A \subset X$  such that  $A/G$  is compact. However, because a Dirac operator on a  $G$ -manifold  $M$  does not naturally define a class in  $RK_0^G(M)$ , we will always assume that the orbit spaces of the actions we consider are compact.

The assembly map was introduced to state the *Baum–Connes conjecture*. This conjecture states that if  $\underline{E}G$  is a classifying space for proper  $G$ -actions (see [8], Sections 1 and 2, and Appendix 1), then the assembly map

$$\mu_{\underline{E}G}^G : RK_0^G(\underline{E}G) \rightarrow K_0(C_r^*(G))$$

is an isomorphism of abelian groups. More on the Baum–Connes conjecture can be found in [8, 61, 80]. A proof for groups with finitely many connected components is given in [15].

### The assembly map in the compact setting

The reason why the assembly map can be interpreted as a generalised equivariant index is the following fact.

**Proposition 5.17.** *Let  $M$  be a compact manifold, on which a compact group  $K$  acts properly. Let  $D$  be a first order elliptic differential operator on  $M$  as in Theorem 4.33, so that we have the class  $[D] \in K_0^K(M)$ . Then*

$$\mu_M^K[D] = K\text{-index } D \in R(K) \cong K_0(C^*(K)).$$

*Sketch of proof.* Let  $\text{pt}$  be the one-point space, and consider the map  $p : K_0^K(M) \rightarrow K_0^K(\text{pt})$  induced by collapsing  $M$  to a point:

$$p[\mathcal{H}, F, \pi] = [\mathcal{H}, F],$$

where on the right hand side, the representation of  $C(\text{pt}) = \mathbb{C}$  on  $\mathcal{H}$  is given by scalar multiplication. It follows directly from the definition of the assembly map, and from compactness of  $M$ , that the following diagram commutes:

$$\begin{array}{ccc} K_0^K(M) & \xrightarrow{\mu_M^K} & K_0(C^*(K)). \\ p \downarrow & \nearrow \mu_{\text{pt}}^K & \\ K_0^K(\text{pt}) & & \end{array} \quad (5.5)$$

Now since  $K_0^K(\text{pt}) \cong R(K)$  via the index map, it can be shown that

$$p[D] = K\text{-index } D \in R(K),$$

for all  $K$ -homology classes  $[D] \in K_0^K(M)$  as in the statement of the proposition. Furthermore, it turns out that  $\mu_{\text{pt}}^K$  is the isomorphism  $R(K) \cong K_0^K(\text{pt}) \cong K_0(C^*(K))$  described above Proposition 4.29. Therefore, the proposition follows from commutativity of diagram (5.5).  $\square$

*Sketch of an alternative proof.* An alternative proof of Proposition 5.17 is based on an explicit description of the assembly map in the compact case. Indeed, by Proposition 4.7, we have  $C^*(K) \cong \bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi)$ . For every irreducible (unitary) representation  $(V_\pi, \pi)$  of  $K$ , and with  $M$ ,  $E$ ,  $D$  and  $K = G$  as in Theorem 4.33, let  $E_\pi \rightarrow M/K$  be the vector bundle

$$E_\pi := (E \otimes \mathcal{B}(V_\pi))/K.$$

Here  $K$  acts on  $E \otimes \mathcal{B}(V_\pi)$  by  $k \cdot (e \otimes a) = k \cdot e \otimes a \circ k^{-1}$ , for all  $k \in K$ ,  $e \in E$  and  $a \in \mathcal{B}(V_\pi)$ . The  $K$ -equivariant operator  $D$  on  $\Gamma^\infty(E)$  naturally induces an operator  $D_\pi$  on  $\Gamma^\infty(E_\pi)$ , which acts trivially on  $\mathcal{B}(V_\pi)$ .

Let  $\bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$  be the completion of the algebraic direct sum in the  $\bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi)$ -valued inner product given by

$$(s_\pi^1 \otimes \varphi_\pi^1, s_\pi^2 \otimes \varphi_\pi^2) = \int_M (s_\pi^1(m), s_\pi^2(m))_E (\varphi_\pi^1(m))^* \varphi_\pi^2(m) dm,$$

for  $s_\pi^j \in L^2(E)$  and  $\varphi_\pi^j \in L^2(M, \mathcal{B}(V_\pi))$  such that  $s_\pi^j \otimes \varphi_\pi^j \in L^2(E \otimes \mathcal{B}(V_\pi))^K \cong L^2(E_\pi)$ . The resulting norm on  $\bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$  is explicitly given by

$$\left\| \bigoplus_{\pi \in \hat{K}} s_\pi \right\| = \sup_{\pi \in \hat{K}} \|s_\pi\|_{L^2(E_\pi)},$$

for  $s_\pi \in L^2(E_\pi)$ . In this way,  $\bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$  becomes a Hilbert  $\bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi)$ -module, and we claim that

$$\mu_M^K[D] = \left[ \bigoplus_{\pi \in \hat{K}} L^2(E_\pi), \bigoplus_{\pi \in \hat{K}} b(D_\pi) \right] \in K_0 \left( \bigoplus_{\pi \in \hat{K}} \mathcal{B}(V_\pi) \right) \cong K_0(C^*(K)), \quad (5.6)$$

where  $b$  is a normalising function.

The equality (5.6) follows from the fact that the map

$$T : L^2(E) = L^2(E)_c \rightarrow \bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$$

given by

$$(Ts)(Km)v = \int_K k \cdot s(k^{-1}m) \otimes k \cdot v dk,$$

for all  $s \in L^2(E)$  and  $v \in V_\pi$ , extends to an isomorphism  $\mathcal{E} \cong \bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$  of Hilbert  $C^*(K)$ -modules, which intertwines the operators  $b(D)_\mathcal{E}$  on  $\mathcal{E}$  and  $\bigoplus_{\pi \in \hat{K}} b(D_\pi)$  on  $\bigoplus_{\pi \in \hat{K}} L^2(E_\pi)$ .

To finish the proof of Proposition 5.17, one shows that the class (5.6) is mapped to the class

$$\bigoplus_{\pi \in \hat{K}} [\ker D_\pi^+] - [\ker D_\pi^-] \in R(K),$$

which equals

$$\begin{aligned} & \bigoplus_{\pi \in \hat{K}} [(\ker D^+ \otimes \mathcal{B}(V_\pi))^K] - [(\ker D^- \otimes \mathcal{B}(V_\pi))^K] = \\ & \bigoplus_{\pi \in \hat{K}} [(\ker D^+ \otimes V_\pi^*)^K \otimes V_\pi] - [(\ker D^- \otimes V_\pi^*)^K \otimes V_\pi] = [\ker D^+] - [\ker D^-], \end{aligned}$$

by Schur's lemma. □

Note that the ‘index’-aspect of the assembly map, by which we mean taking a kernel and a cokernel, lies in the isomorphisms  $KK_0(\mathbb{C}, C^*(K)) \cong K_0(C^*(K)) \cong R(K)$  of Theorem 5.12 and Proposition 4.29, and not in the actual definition of the assembly map itself.

Because of Proposition 5.17, we will see that Definitions 6.1 and 6.2 of quantisation reduce to Definitions 3.20 and 3.30 in the compact case.

## 5.3 The unbounded picture of $KK$ -theory

In [7], Baaj and Julg developed a realisation of  $KK$ -theory using unbounded operators instead of bounded ones. The advantage of this realisation is that the Kasparov product has a simpler form in this setting. We will use this form in the proof of Theorem 9.3. The intuitive idea is that the unbounded Kasparov bimodules introduced by Baaj and Julg are generalisations of first order elliptic pseudo-differential operators, whereas the bounded Kasparov bimodules of Definition 5.5 generalise elliptic pseudo-differential operators of order zero.

## Unbounded $KK$ -theory

**Definition 5.18.** Let  $A$  and  $B$  be  $C^*$ -algebras. An *unbounded  $\mathbb{Z}_2$ -graded Kasparov  $(A, B)$ -bimodule* is a triple  $(\mathcal{E}, D, \pi)$ , where  $\mathcal{E}$  and  $\pi$  are as in Definition 5.8 (without the group  $G$ ), and  $D$  is a self-adjoint unbounded operator<sup>2</sup> on  $\mathcal{E}$  that reverses the grading on  $\mathcal{E}$ , and has the following properties.

- $D$  is *regular*, in the sense that the image of  $1_{\mathcal{E}} + D^2$  is dense in  $\mathcal{E}$ ;
- for all  $a \in A$ , the operator  $\pi(a)(1 + D^2)^{-1}$  is compact;
- the set of  $a \in A$  such that the graded commutator  $[D, \pi(a)]$  is well-defined on  $\text{dom } D$  and extends continuously to an adjointable operator on  $\mathcal{E}$ , is dense in  $A$ .

The set of unbounded  $\mathbb{Z}_2$ -graded Kasparov  $(A, B)$ -bimodules is denoted by  $\Psi_0(A, B)$ .

The central result in unbounded  $KK$ -theory is the following (see [7], Proposition 2.3).

**Theorem 5.19.** *The map*

$$\beta : \Psi_0(A, B) \rightarrow KK_0(A, B)$$

*defined by*

$$\beta(\mathcal{E}, D, \pi) = \left[ \mathcal{E}, \frac{D}{\sqrt{1 + D^2}}, \pi \right]$$

*is a well-defined surjection.*

## The unbounded Kasparov product

Now, for  $j = 1, 2$ , let  $A_j$  and  $B_j$  be  $C^*$ -algebras. Suppose that the algebras  $A_j$  are separable, and that the  $B_j$  are  $\sigma$ -unital. In the special case where  $C = \mathbb{C}$ , the Kasparov product (5.3) has the following description in terms of unbounded Kasparov bimodules.

Let  $(\mathcal{E}_j, D_j, \pi_j) \in \Psi_0(A_j, B_j)$  be given. Let  $D$  be the closure of the operator  $D_1 \otimes 1_{\mathcal{E}_2} + 1_{\mathcal{E}_1} \otimes D_2$  on  $\mathcal{E}_1 \otimes \mathcal{E}_2$ . Then define

$$(\mathcal{E}_1, D_1, \pi_1) \times (\mathcal{E}_2, D_2, \pi_2) := (\mathcal{E}_1 \otimes \mathcal{E}_2, D, \pi_1 \otimes \pi_2).$$

**Theorem 5.20.** *This is an element of  $\Psi_0(A_1 \otimes A_2, B_1 \otimes B_2)$ , and the following diagram commutes:*

$$\begin{array}{ccc} \Psi_0(A_1, B_1) \times \Psi_0(A_2, B_2) & \xrightarrow{\times} & \Psi_0(A_1 \otimes A_2, B_1 \otimes B_2) \\ \beta \times \beta \downarrow & & \downarrow \beta \\ KK_0(A_1, B_1) \times KK_0(A_2, B_2) & \xrightarrow{\times} & KK_0(A_1 \otimes A_2, B_1 \otimes B_2). \end{array}$$

See [7], Theorem 3.2.

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<sup>2</sup>Self-adjoint unbounded operators on Hilbert modules over  $C^*$  algebras are defined analogously to such operators on Hilbert spaces (see Section 4.3).

**Remark 5.21** (Equivariant unbounded  $KK$ -theory). There is an equivariant version of unbounded  $KK$ -theory. The operators in equivariant unbounded Kasparov bimodules are supposed to satisfy a condition that is much weaker than equivariance with respect to the given group actions. We will only use equivariant unbounded  $K$ -homology of topological spaces however (that is,  $A_1$  and  $A_2$  are commutative, and  $B_1 = B_2 = \mathbb{C}$ ). In that case it suffices to consider unbounded Kasparov bimodules with strictly equivariant operators, by Lemma 5.14.

## The assembly map

Next, we describe the analytic assembly map in the unbounded picture of  $KK$ -theory. We will use this description in the proof of Theorem 9.3.

For full group  $C^*$ -algebras, the assembly map in the unbounded picture is defined in Kucerovsky's appendix to [61], in the following way. Let  $G$  be a second countable, locally compact Hausdorff group, acting properly on a locally compact Hausdorff space  $X$ , with compact quotient. The assembly map in the unbounded picture is given by

$$\mu_X^G(\mathcal{H}, D, \pi) = (\mathcal{E}, D_{\mathcal{E}}) \in \Psi_0(\mathbb{C}, C^*G), \quad (5.7)$$

for all  $(\mathcal{H}, D, \pi) \in \Psi_0^G(C_0(X), \mathbb{C})$ . The Hilbert  $C^*(G)$ -module  $\mathcal{E}$  is defined as usual for the assembly map. The definition of the operator  $D_{\mathcal{E}}$  on  $\mathcal{E}$  is more involved.

First, let  $\tilde{\mathcal{H}}$  be the auxiliary Hilbert  $C^*(G)$ -module defined as the completion of the Hilbert  $C_c(G)$ -module  $C_c(G, \mathcal{H})$  with respect to the  $C_c(G) \subset C^*(G)$ -valued inner product

$$(\varphi, \psi)_{\tilde{\mathcal{H}}}(g) := \int_G (\varphi(g'), \psi(g'g))_{\mathcal{H}} dg', \quad (5.8)$$

where  $\varphi, \psi \in C_c(G, \mathcal{H})$ ,  $g \in G$ , and  $dg'$  is a Haar measure on  $G$ . Next, let  $h \in C_c(X)$  be a function such that for all  $x \in X$ ,

$$\int_G h^2(gx) dg = 1$$

(see Lemma 7.8).

Let  $p \in C_c(X \times G)$  be the projection given by

$$p(x, g) := \overline{h(x)} h(g^{-1}x). \quad (5.9)$$

This function is compactly supported by properness of the action of  $G$  on  $X$ . Let  $\tilde{\pi} : C_c(X \times G) \rightarrow \mathcal{B}(\tilde{\mathcal{H}})$  be the representation given by

$$(\tilde{\pi}(f)\varphi)(g) = \int_G \pi(f(-, g')) g' \cdot \varphi(g'^{-1}g) dg',$$

for  $f \in C_c(X \times G)$ ,  $\varphi \in C_c(G, \mathcal{H})$  and  $g \in G$ . (The representation  $\tilde{\pi}$  can actually be extended to the crossed product  $C_0(X) \rtimes G$ , but we will not use this extension.)

Then the map

$$\alpha : \tilde{\pi}(p)C_c(G, \mathcal{H}) \rightarrow \mathcal{H}_c,$$

given by

$$\tilde{\pi}(p)\varphi \mapsto \int_G g^{-1} \pi(h)\varphi(g) dg,$$

preserves the  $C^*(G)$ -valued inner products and the  $C^*(G)$ -module structures on  $\mathcal{H}$  and on  $\mathcal{E}$ , and induces an isomorphism  $\tilde{\pi}(p)\mathcal{H} \cong \mathcal{E}$  of Hilbert  $C^*(G)$ -modules. We will write  $\tilde{\mathcal{E}} := \tilde{\pi}(p)\mathcal{H}$ .

To define the operator  $D_{\mathcal{E}}$  on  $\mathcal{E}$  we first consider an operator  $D_{\tilde{\mathcal{E}}}$  on  $\tilde{\mathcal{E}}$ . This operator is defined as the closure of the operator  $\tilde{D}$  on  $\tilde{\mathcal{E}}$ , given by

$$\tilde{D}(\tilde{\pi}(p)\varphi) := \tilde{\pi}(p)(D \circ \varphi), \quad (5.10)$$

on the domain  $\text{dom } \tilde{D} := \tilde{\pi}(p)C_c(G, \text{dom } D)$ . We finally set

$$D_{\mathcal{E}} := \alpha D_{\tilde{\mathcal{E}}} \alpha^{-1},$$

on the domain  $\text{dom } D_{\mathcal{E}} = \alpha(\text{dom } D_{\tilde{\mathcal{E}}})$ .

In the proof of Theorem 9.3, we will actually use the following definition of the assembly map:

$$\tilde{\mu}_X^G(\mathcal{H}, D, \pi) := (\tilde{\mathcal{E}}, D_{\tilde{\mathcal{E}}}) \in \Psi_0(\mathbb{C}, C^*G), \quad (5.11)$$

which gives the same class in  $K_0(C^*(G))$  as (5.7), because  $\alpha$  is an isomorphism.

Kucerovsky's proof that the above constructions give a well-defined description of the assembly map in the unbounded picture is valid for discrete groups, but it admits a straightforward generalisation to possibly nondiscrete (unimodular) ones. One simply replaces sums by integrals, and uses the fact that the integral over a compact, finite Borel space of a continuous family of adjointable operators is again an adjointable operator (see Lemma 7.2). In addition, in the proof of Lemma 2.15 in [61], one takes  $\beta^{-1}(\pi(f)\eta) = \tilde{\pi}(p)\psi$ , with  $\psi(g) = \pi(h)\pi(g \cdot f)g \cdot \eta$  (where the  $\beta$  in [61] is our  $\alpha$ ). This reduces to Valette's  $\beta^{-1}(\pi(f)\eta) = \tilde{\pi}(p)\tilde{\pi}(\langle h|f \rangle)\tilde{\eta}$  in the discrete case.

To use the unbounded picture of the assembly map for *reduced* group  $C^*$ -algebras, one can use the above description for the full  $C^*$ -algebra, use the map  $\beta$  to descend to  $KK$ -theory, and then apply the map  $\lambda_G$  (see Remark 5.16).

# Chapter 6

## Noncommutative geometry and quantisation: statement of the results

In this chapter, we state the two main results of this thesis. Using the techniques from Chapters 4 and 5, we extend the Guillemin–Sternberg conjecture, Theorem 3.34, to noncompact groups and manifolds. To state this generalisation, we replace the index by the assembly map. The assumptions that the group and the manifold in question are compact are replaced by the assumption that the quotient space of the action is compact, i.e. that the action is cocompact.

We first state a generalisation of Theorem 3.34 to cocompact Hamiltonian actions by any Lie group. This generalisation, Conjecture 6.4, was formulated by Landsman in [50], and is the subject of Section 6.1. We will prove a special case of this conjecture, Theorem 6.5, in Part III.

In Section 6.3, we state a generalisation of Theorem 3.38 to cocompact Hamiltonian actions by *semisimple* Lie groups. This generalisation, Theorem 6.13, is based on V. Lafforgue’s work on discrete series representations in the context of the  $K$ -theory of reduced group  $C^*$ -algebras, which is summarised in Section 6.2. In Part IV, we prove Theorem 6.13.

### 6.1 Quantisation commutes with reduction for cocompact group actions

Let  $(M, \omega)$  be a symplectic manifold. Let  $G$  be a Lie group acting properly and in Hamiltonian fashion on  $(M, \omega)$ , with momentum map  $\Phi$ . Suppose that  $M/G$  is compact.

#### Quantisation of cocompact actions

We first generalise Dolbeault-quantisation to the cocompact case. Let  $J$  be a  $G$ -equivariant almost complex structure on  $M$ , compatible with  $\omega$ . Such a  $J$  always exists, by [27], Example D.12 and Corollary B.35. Let  $g := \omega(-, J-)$  be the associated Riemannian metric on  $M$ . Suppose that there is a  $G$ -equivariant prequantisation  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  of the action of  $G$  on  $(M, \omega)$  (see Remark 3.9).

Let  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  be the Dolbeault–Dirac operator on the vector bundle  $\wedge^{0,*} T^*M \otimes L^\omega$  (Definition 3.19). It defines a class  $[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*] \in K_0^G(M)$  by Corollary 4.36. This class is independent of the connection  $\nabla$  and the choice of  $J$ .



**Definition 6.1** (Quantisation V, Landsman [50]). The *Dolbeault-quantisation* of the action of  $G$  on  $(M, \omega)$  is the  $K$ -theory class

$$Q_V(M, \omega) := \mu_M^G[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*] \in K_0(C^*(G)).$$

The definition of  $\text{Spin}^c$ -quantisation can be generalised in a similar way. Let  $(L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla)$  be a  $G$ -equivariant  $\text{Spin}^c$ -quantisation of  $(M, \omega)$ , and let  $P \rightarrow M$  be an equivariant  $\text{Spin}^c$ -structure on  $M$  with determinant line bundle  $L^{2\omega}$ . Let  $\not{D}_M^{L^\omega}$  be the  $\text{Spin}^c$ -Dirac operator on the associated spinor bundle (Definition 3.27). Then we have the  $K$ -homology class  $[\not{D}_M^{L^\omega}] \in K_0^G(M)$ , by Corollary 4.36.

**Definition 6.2** (Quantisation VI). The  $\text{Spin}^c$ -quantisation of the action of  $G$  on  $(M, \omega)$  is the  $K$ -theory class

$$Q_{VI}(M, \omega) := \mu_M^G[\not{D}_M^{L^\omega}] \in K_0(C_r^*(G)).$$

Note that we now use the reduced  $C^*$ -algebra of  $G$ , instead of the full one used in Definition 6.1. The reason for this difference is that we will use Definition 6.1 to state a ‘quantisation commutes with reduction’-result for reduction at the trivial representation, which implies that we have to use the full group  $C^*$ -algebra. We will use Definition 6.2 to state a ‘quantisation commutes with reduction’-result for reduction at discrete series representations of semisimple Lie groups, and in that case, it is more natural to work with the reduced group  $C^*$ -algebra. This choice between the full and the reduced  $C^*$ -algebra is not at all related to the difference between Dolbeault-quantisation and  $\text{Spin}^c$ -quantisation, and both types of quantisation can be defined using the full or reduced group  $C^*$ -algebra.

**Remark 6.3.** Now that we have given the sixth and last definition of geometric quantisation, let us summarise the relations between these definitions.

- If  $M$  and  $G$  are *compact*, then we have

$$\begin{aligned} Q_V(M, \omega) &= Q_{III}(M, \omega); \\ Q_{VI}(M, \omega) &= Q_{IV}(M, \omega) \end{aligned}$$

(see Proposition 5.17).

- If the line bundle  $\bigwedge_{\mathbb{C}}^{0, d_M}(TM, J)$  is trivial for some equivariant almost complex structure  $J$ , compatible with  $\omega$ , then

$$Q_{VI}(M, \omega) = Q_V(M, \omega),$$

and if, in addition,  $M$  and  $G$  are compact, then

$$Q_{IV}(M, \omega) = Q_{III}(M, \omega)$$

(see Lemma 3.32).

- If  $M$  and  $G$  are compact, and  $(M, \omega)$  is *Kähler*, then

$$Q_{III}(M, \omega) = Q_{II}(M, \omega)$$

(see Lemma 3.23).

- If  $M$  and  $G$  are compact,  $(M, \omega)$  is Kähler, and  $\omega$  is *positive*, then

$$Q_{II}(M, n\omega) = Q_I(M, n\omega),$$

for  $n$  large enough (see Remark 3.16).

We will only use  $Q_V$  and  $Q_{VI}$  from now on.

## Reduction

The reduction map

$$R_G^0 : K_0(C^*(G)) \rightarrow \mathbb{Z} \quad (6.1)$$

that generalises taking the multiplicity of the trivial representation as in (3.15), is defined as follows. The map

$$\int_G : C_c(G) \rightarrow \mathbb{C} \quad (6.2)$$

given by

$$\int_G(f) = \int_G f(g) dg$$

(with  $dg$  a Haar measure) is the one associated to the trivial representation of  $G$ . It is continuous with respect to the norm  $\|\cdot\|_{C^*(G)}$  on  $C_c(G)$ . Because the trivial representation is not contained in  $L^2(G)$  for noncompact  $G$ , the map (6.2) is not continuous with respect to the norm on the *reduced* group  $C^*$ -algebra of  $G$  in the noncompact case. This is why we work with the full one here.

The continuous extension of (6.2) to a map  $C^*(G) \rightarrow \mathbb{C}$  induces a map on  $K$ -theory

$$R_G^0 := (\int_G)_* : K_0(C^*(G)) \rightarrow K_0(\mathbb{C}) \cong \mathbb{Z} \quad (6.3)$$

Using the fact that the constant function 1 on  $G$  is in  $C_c(G) \subset C^*(G)$  if  $G$  is compact, one can show that the map  $R_G^0$  is given by (3.15) for compact  $G = K$ .

Since  $M/G$  is compact, the symplectic reduction  $M_0 = \Phi^{-1}(0)/G$  is compact as well. Suppose that 0 is a regular value of  $\Phi$ . Then the quantisation  $Q_{III}(M_0, \omega_0)$  is well-defined (see Section 3.6). Here we use  $Q_{III}$  instead of  $Q_V$ , since  $Q_{III}(M_0, \omega_0) = Q_V(M_0, \omega_0)$  if  $M_0$  is smooth, and we do not know if  $Q_V(M_0, \omega_0)$  is well-defined if  $M_0$  is an orbifold. This would depend on an orbifold version of Corollary 4.36.

We now have all ingredients needed to state the following conjecture.

**Conjecture 6.4** (Guillemin–Sternberg–Landsman conjecture). *If  $0 \in \Phi(M)$ , then the following integers are equal:*

$$\boxed{R_G^0(Q_V(M, \omega)) := (\int_G)_* (\mu_M^G [\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*]) = Q_{III}(M_0, \omega_0).}$$

*If  $0 \notin \Phi(M)$ , then  $R_G^0(Q_V(M, \omega)) = 0$ .*

In [50], Landsman states Conjecture 6.4 as a special case of a more far-reaching conjecture called ‘functoriality of quantisation’. The latter conjecture states that quantisation can be defined as a functor between the category of Poisson manifolds, with Weinstein dual pairs as

arrows, and the category of  $C^*$ -algebras, with  $KK$ -groups as sets of arrows. The object part of this conjectural quantisation functor should be defined by deformation quantisation, whereas the arrow part should be given by geometric quantisation.

A subgroup  $H < G$  is called *cocompact* if  $G/H$  is compact. In Part III, we prove the following result:

**Theorem 6.5.** *Suppose  $G$  has a cocompact, discrete, normal subgroup  $\Gamma \triangleleft G$ . Suppose furthermore that  $\Gamma$  acts freely on  $M$ . Finally, assume that  $M$  is complete<sup>1</sup> in the Riemannian metric  $g$ . With these additional assumptions, Conjecture 6.4 is true.*

In the setting of Theorem 6.5, we will denote the compact group  $G/\Gamma$  by  $K$ . Examples of groups  $G$  that satisfy the assumptions of Theorem 6.5 are:

- $G = K$  is compact, and  $\Gamma = \{e_G\}$ ;
- $G = \Gamma$  is discrete, and  $K = \{e_K\}$ ;
- $G = \mathbb{R}^n$ ,  $\Gamma = \mathbb{Z}^n$  and  $K = \mathbb{T}^n$  for some  $n \in \mathbb{N}$ ,

or direct products of these three examples. In fact, if  $G$  is connected, then  $\Gamma$  must be central, and  $G$  is the direct product of a compact group and a vector space.

**Remark 6.6.** One can try to make life easier by assuming that the action of  $G$  on  $M$  is free. However, in the situation of Theorem 6.5, this assumption implies that  $G$  is *discrete*.

Indeed, if the action is locally free then by Smale's lemma (Lemma 2.24), the momentum map  $\Phi$  is a submersion, and in particular an open mapping. And since it is  $G$ -equivariant, it induces

$$\Phi^G : M/G \rightarrow \mathfrak{g}^* / \text{Ad}^*(G),$$

which is also open. So, since  $M/G$  is compact, the image

$$\Phi^G(M/G) \subset \mathfrak{g}^* / \text{Ad}^*(G)$$

is a compact open subset. Because  $\mathfrak{g}^* / \text{Ad}^*(G)$  is connected,<sup>2</sup> it must therefore be compact. This, however, can only be the case (under the assumptions of Theorem 6.5) when  $G$  is discrete. Indeed, we have

$$\text{Ad}^*(G) \cong \text{Ad}^*(K) \subset \text{GL}(\mathfrak{k}^*) \cong \text{GL}(\mathfrak{g}^*).$$

So  $\text{Ad}^*(G)$  is compact, and  $\mathfrak{g}^* / \text{Ad}^*(G)$  cannot be compact, unless  $\mathfrak{g}^* = 0$ , i.e.  $G$  is discrete.

**Example 6.7.** Suppose  $(M_1, \omega_1)$  is a compact symplectic manifold,  $K$  is a compact Lie group, and let a proper Hamiltonian action of  $K$  on  $M_1$  be given. Suppose that  $(M_1, \omega_1)$  has an equivariant prequantisation. Let  $\Gamma$  be a discrete group acting properly and freely on a symplectic manifold  $(M_2, \omega_2)$ , leaving  $\omega_2$  invariant. Suppose that  $M_2/\Gamma$  is compact, and that there is an equivariant prequantisation of  $(M_2, \omega_2)$ . Then the direct product action of  $K \times \Gamma$  on  $M_1 \times M_2$  satisfies the assumptions of Theorem 6.5.

**Remark 6.8.** In the case where  $G$  is a torsion-free discrete group acting freely on  $M$ , Theorem 6.5 follows from a result of Pierrot ([67], Théorème 3.3.2).

<sup>1</sup>see Remark 4.35

<sup>2</sup>If  $G = K$  is a compact connected Lie group, then  $\mathfrak{k}^* / \text{Ad}^*(K)$  is a Weyl chamber.

### A refinement?

To state a more refined version of Conjecture 6.4, which includes reduction at more representations than just the trivial one, we need an ‘orbit method’ for the group  $G$ . The orbit method is an idea of Kirillov [42, 43, 44]. It is an attempt to realise irreducible unitary representations  $\mathcal{H}$  as quantisations  $\mathcal{H} = \mathcal{H}_\theta$  of coadjoint orbits  $\theta \subset \mathfrak{g}^*$  (see Example 2.13) in a subset  $A \subset \mathfrak{g}^*$ .

The symplectic reduction of  $M$  at a coadjoint orbit  $\theta$  can be defined as  $M_\theta := \Phi^{-1}(\theta)/G$ . If all irreducible representations  $\mathcal{H}_\theta$  define classes  $[\mathcal{H}_\theta] \in K_0(C^*(G))$ , then we can try to make sense of the following statement:

$$“\mu_M^G[\mathcal{D}_M^L] = \bigoplus_{\theta \subset A} Q(M_\theta, L_\theta)[\mathcal{H}_\theta]”.$$
 (6.4)

Or, if  $R_G^\theta : K_0(C^*(G)) \rightarrow \mathbb{Z}$  is a suitable reduction map,

$$“R_G^\theta(\mu_M^G[\mathcal{D}_M^L]) = Q(M_\theta, L_\theta)”.$$
 (6.5)

For compact groups, the appropriate orbit method is the Borel–Weil theorem (Example 3.36). For discrete series representations, the ‘orbit method’ we will use is described in Section 6.2, although this method does not use coadjoint orbits, but other homogeneous spaces. The resulting version of (6.5) is Theorem 6.13, which is stated using  $\text{Spin}^c$ -quantisation instead of Dolbeault-quantisation. We will prove this result in Part IV.

A final note is that the decomposition (6.4) only makes sense if the set  $A/G$  is *discrete*. Otherwise, the direct sum would have to be replaced by a direct integral with respect to a suitable measure on  $A/G$ . The author has no idea how to state a ‘quantisation commutes with reduction’ theorem in this situation. In any case, this shows that it is natural to restrict one’s attention to discrete series representations of a semisimple group when trying to state (6.4) rigorously for such groups.

## 6.2 Discrete series representations and $K$ -theory

In [48], V. Lafforgue reproves some classical results about discrete series representations by Harish-Chandra [30, 31], Atiyah & Schmid [5] and Parthasarathy [65], using  $K$ -homology,  $K$ -theory and assembly maps. We will give a quick summary of the results in [48] that we will use in this thesis.

For the remainder of this chapter, let  $G$  be a connected<sup>3</sup> semisimple Lie group with finite centre. Let  $K < G$  be a maximal compact subgroup, and let  $T < K$  be a maximal torus. Suppose that  $T$  is also a Cartan subgroup of  $G$ , so that  $G$  has discrete series representations by Harish-Chandra’s criterion [31]. Discrete series representations are representations whose matrix elements are square-integrable over  $G$ . They form a discrete subset of the unitary dual of  $G$ .

In [65], Parthasarathy realises the irreducible discrete series representations of  $G$  as the  $L^2$ -indices of Dirac operators  $\mathcal{D}^V$ , where  $V$  runs over the irreducible representations of  $K$ . Atiyah

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<sup>3</sup>Theorem 6.13 and the results in this Part IV (possibly in modified forms) are also valid for groups with finitely many connected components, but the assumption that  $G$  is connected allows us to circumvent some technical difficulties.

and Schmid do the same in [5], replacing Harish-Chandra's work by results from index theory. In [71, 72, 73], Slebarsky considers the decomposition into irreducible representations of  $G$  of  $L^2$ -indices of Dirac operators on any homogeneous space  $G/L$ , with  $L < G$  a compact, connected subgroup.

## Dirac induction

For a given irreducible representation  $V$  of  $K$ , the Dirac operator  $\mathcal{D}^V$  used by Parthasarathy and Atiyah–Schmid is defined as follows. Let  $\mathfrak{p} \subset \mathfrak{g}$  be the orthogonal complement to  $\mathfrak{k}$  with respect to the Killing form. Then  $\mathfrak{p}$  is an  $\text{Ad}(K)$ -invariant linear subspace of  $\mathfrak{g}$ , and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Consider the inner product on  $\mathfrak{p}$  given by the restriction of the Killing form. The adjoint representation

$$\text{Ad} : K \rightarrow \text{GL}(\mathfrak{p})$$

of  $K$  on  $\mathfrak{p}$  takes values in  $\text{SO}(\mathfrak{p})$ , because the Killing form is  $\text{Ad}(K)$ -invariant, and  $K$  is connected. We suppose that it has a lift  $\widetilde{\text{Ad}}$  to the double cover  $\text{Spin}(\mathfrak{p})$  of  $\text{SO}(\mathfrak{p})$ . It may be necessary to replace  $G$  and  $K$  by double covers for this lift to exist. Then the homogeneous space  $G/K$  has a  $G$ -equivariant Spin-structure

$$P^{G/K} := G \times_K \text{Spin}(\mathfrak{p}) \rightarrow G/K.$$

Here  $G \times_K \text{Spin}(\mathfrak{p})$  is the quotient of  $G \times \text{Spin}(\mathfrak{p})$  by the action of  $K$  defined by

$$k(g, a) = (gk^{-1}, \widetilde{\text{Ad}}(k)a),$$

for  $k \in K$ ,  $g \in G$  and  $a \in \text{Spin}(\mathfrak{p})$ .

Fix an orthonormal basis  $\{X_1, \dots, X_{d_{\mathfrak{p}}}\}$  of  $\mathfrak{p}$ . Using this basis, we identify  $\text{Spin}(\mathfrak{p}) \cong \text{Spin}(d_{\mathfrak{p}})$ . Let  $\Delta_{d_{\mathfrak{p}}}$  be the canonical  $2^{\frac{d_{\mathfrak{p}}}{2}}$ -dimensional representation of  $\text{Spin}(d_{\mathfrak{p}})$  (see Section 3.4). Because  $\mathfrak{p}$  is even-dimensional,  $\Delta_{d_{\mathfrak{p}}}$  splits into two irreducible subrepresentations  $\Delta_{d_{\mathfrak{p}}}^+$  and  $\Delta_{d_{\mathfrak{p}}}^-$ . Consider the  $G$ -vector bundles

$$E_V^{\pm} := G \times_K (\Delta_{d_{\mathfrak{p}}}^{\pm} \otimes V) \rightarrow G/K.$$

Note that

$$\Gamma^{\infty}(G/K, E_V^{\pm}) \cong (C^{\infty}(G) \otimes \Delta_{d_{\mathfrak{p}}}^{\pm} \otimes V)^K, \quad (6.6)$$

where  $K$  acts on  $C^{\infty}(G) \otimes \Delta_{d_{\mathfrak{p}}}^{\pm} \otimes V$  by

$$k \cdot (f \otimes \delta \otimes v) = (f \circ l_{k^{-1}} \otimes \widetilde{\text{Ad}}(k)\delta \otimes k \cdot v) \quad (6.7)$$

for all  $k \in K$ ,  $f \in C^{\infty}(G)$ ,  $\delta \in \Delta_{d_{\mathfrak{p}}}$  and  $v \in V$ . Here  $l_{k^{-1}}$  denotes left multiplication by  $k^{-1}$ .

Using the basis  $\{X_1, \dots, X_{d_{\mathfrak{p}}}\}$  of  $\mathfrak{p}$  and the isomorphism (6.6), define the differential operator

$$\mathcal{D}^V : \Gamma^{\infty}(E_V^+) \rightarrow \Gamma^{\infty}(E_V^-) \quad (6.8)$$

by the formula

$$\mathcal{D}^V := \sum_{j=1}^{d_{\mathfrak{p}}} X_j \otimes c(X_j) \otimes 1_V. \quad (6.9)$$

Here in the first factor,  $X_j$  is viewed as a left invariant vector field on  $G$ , and in the second factor,  $c : \mathfrak{p} \rightarrow \text{End}(\Delta_{d_p})$  is the Clifford action (see Section 3.4). This action is odd with respect to the grading on  $\Delta_{d_p}$ . The operator (6.8) is the Spin-Dirac operator on  $G/K$  (see [65], Proposition 1.1 and [22], Chapter 3.5).

Lafforgue (see also Wassermann [86]) uses the same operator to define a ‘Dirac induction map’

$$\text{D-Ind}_K^G : R(K) \rightarrow K_0(C_r^*(G)) \quad (6.10)$$

by

$$\text{D-Ind}_K^G[V] := \left[ (C_r^*(G) \otimes \Delta_{d_p} \otimes V)^K, b(\not{D}^V) \right], \quad (6.11)$$

where  $b : \mathbb{R} \rightarrow \mathbb{R}$  is a normalising function, e.g.  $b(x) = \frac{x}{\sqrt{1+x^2}}$ . The expression on the right hand side defines a class in Kasparov’s  $KK$ -group  $KK_0(\mathbb{C}, C_r^*(G))$ , which is isomorphic to the  $K$ -theory group  $K_0(C_r^*(G))$  by Theorem 5.12. In [86], Wassermann proves the Connes–Kasparov conjecture, which states that this Dirac induction map is a bijection for linear reductive groups.

## Reduction

The relation between the Dirac induction map and the work of Atiyah & Schmid and of Parthasarathy can be seen by embedding the discrete series of  $G$  into  $K_0(C_r^*(G))$  via the map

$$\mathcal{H} \mapsto [\mathcal{H}] := [d_{\mathcal{H}} c_{\mathcal{H}}],$$

where  $\mathcal{H}$  is a Hilbert space with inner product  $(-, -)_{\mathcal{H}}$ , equipped with a discrete series representation of  $G$ ,  $c_{\mathcal{H}} \in C(G)$  is the function

$$c_{\mathcal{H}}(g) = (\xi, g \cdot \xi)_{\mathcal{H}}$$

(for a fixed  $\xi \in \mathcal{H}$  of norm 1), and  $d_{\mathcal{H}}$  is the inverse of the  $L^2$ -norm of  $c_{\mathcal{H}}$  (so that the function  $d_{\mathcal{H}} c_{\mathcal{H}}$  has  $L^2$ -norm 1). Because  $d_{\mathcal{H}} c_{\mathcal{H}}$  is a projection in  $C_r^*(G)$ , it indeed defines a class in  $K_0(C_r^*(G))$  (see Remark 4.27).

Next, Lafforgue defines a map<sup>4</sup>

$$R_G^{\mathcal{H}} : K_0(C_r^*(G)) \rightarrow \mathbb{Z} \quad (6.12)$$

that amounts to taking the multiplicity of the irreducible discrete series representation  $\mathcal{H}$ , as follows. Consider the map

$$C_r^*(G) \rightarrow \mathcal{K}(\mathcal{H})$$

(the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ ), given on  $C_c(G) \subset C_r^*(G)$  by

$$f \mapsto \int_G f(g) \pi(g) dg. \quad (6.13)$$

Here  $\pi$  is the representation of  $G$  in  $\mathcal{H}$ . Since  $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$ , this map induces a map  $K_0(C_r^*(G)) \rightarrow \mathbb{Z}$  on  $K$ -theory, which by definition is (6.12).

---

<sup>4</sup>In Lafforgue’s notation,  $R_G^{\mathcal{H}}(x) = \langle \mathcal{H}, x \rangle$ .

The map  $R_G^{\mathcal{H}}$  has the property that for all irreducible discrete series representations  $\mathcal{H}$  and  $\mathcal{H}'$  of  $G$ , one has

$$R_G^{\mathcal{H}}([\mathcal{H}']) = \begin{cases} 1 & \text{if } \mathcal{H} \cong \mathcal{H}' \\ 0 & \text{if } \mathcal{H} \not\cong \mathcal{H}'. \end{cases}$$

Hence it can indeed be interpreted as a multiplicity function. For compact groups, it follows from Schur orthogonality that this is indeed the usual multiplicity.

In Section 6.1 we used the full group  $C^*$ -algebra to define reduction at the trivial representation. This is because the trivial representation is not square-integrable for noncompact groups. Indeed, the map (6.2) extends continuously to a function on  $C^*(G)$ , but not to a function on  $C_r^*(G)$ . Now we can use the reduced group  $C^*$ -algebra, since the map (6.13) is continuous with respect to the norm on  $C_r^*(G)$ , for discrete series representations  $\pi$ . It is natural to use the reduced group  $C^*$ -algebra when studying discrete series representations, since they are contained in the left regular representation of  $G$  on  $L^2(G)$ , and the reduced  $C^*$ -algebra is defined in terms of this representation.

Dirac induction links the reduction map  $R_G^{\mathcal{H}}$  to the reduction map 3.17 in the following way.

Let  $R = R(\mathfrak{g}, \mathfrak{t})$  be the root system of  $(\mathfrak{g}, \mathfrak{t})$ , let  $R_c := R(\mathfrak{k}, \mathfrak{t}) \subset R$  be the subset of compact roots, and let  $R_n := R \setminus R_c$  be the set of noncompact roots. Let  $R_c^+ \subset R_c$  be a choice of positive compact roots, and let  $\Lambda_+^{\mathfrak{k}}$  be the set of dominant integral weights of  $(\mathfrak{k}, \mathfrak{t})$  with respect to  $R_c^+$ .

Let  $\mathcal{H}$  be an irreducible discrete series representation of  $G$ . Let  $\lambda$  be the Harish-Chandra parameter of  $\mathcal{H}$  (see [30, 31]) such that  $(\alpha, \lambda) > 0$  for all  $\alpha \in R_c^+$ . Here  $(-, -)$  is a Weyl group invariant inner product on  $\mathfrak{t}_{\mathbb{C}}^*$ . Let  $R^+ \subset R$  be the positive root system defined by

$$\alpha \in R^+ \iff (\alpha, \lambda) > 0,$$

for  $\alpha \in R$ . Then  $R_c^+ \subset R^+$ , and we denote by  $R_n^+ := R^+ \setminus R_c^+$  the set of noncompact positive roots. We will write  $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha$  and  $\rho_c := \frac{1}{2} \sum_{\alpha \in R_c^+} \alpha$ . We will use the fact that  $\lambda - \rho_c$  lies on the dominant weight lattice  $\Lambda_+^{\mathfrak{k}}$ , since  $\lambda \in \Lambda_+^{\mathfrak{k}} + \rho$ .

Note that the dimension of the quotient  $G/K$  equals the number of noncompact roots, which is twice the number of positive noncompact roots, and hence even.

**Lemma 6.9.** *Let  $\mu \in \Lambda_+^{\mathfrak{k}}$  be given. Let  $V_{\mu}$  be the irreducible representation of  $K$  with highest weight  $\mu$ . We have*

$$R_G^{\mathcal{H}}(\text{D-Ind}_K^G[V_{\mu}]) = \begin{cases} (-1)^{\frac{\dim G/K}{2}} & \text{if } \mu = \lambda - \rho_c \\ 0 & \text{otherwise.} \end{cases} \quad (6.14)$$

The relation (6.14) can be summarised as

$$R_G^{\mathcal{H}} \circ \text{D-Ind}_K^G = (-1)^{\frac{\dim G/K}{2}} R_K^{\lambda - \rho_c},$$

with  $R_K^{\lambda - \rho_c}$  as defined below Definition 3.17.

*Proof.* According to Lafforgue [48], Lemma 2.1.1, we have

$$\begin{aligned} R_G^{\mathcal{H}}(\text{D-Ind}_K^G[V_{\mu}]) &= \dim(V_{\mu}^* \otimes \Delta_{d_{\mathfrak{p}}}^* \otimes \mathcal{H})^K \\ &= [\Delta_{d_{\mathfrak{p}}}^* \otimes \mathcal{H}|_K : V_{\mu}], \end{aligned} \quad (6.15)$$

the multiplicity of  $V_\mu$  in  $\Delta_{d_p}^* \otimes \mathcal{H}|_K$ . Let us compute this multiplicity.

By Harish-Chandra's formula (Harish-Chandra [31], Schmid [69], Theorem on page 95/96), the character  $\Theta_\lambda$  of  $\mathcal{H}$  is given by

$$\Theta_\lambda|_{T^{\text{reg}}} = (-1)^{\frac{\dim G/K}{2}} \frac{\sum_{w \in W(\mathfrak{k}, \mathfrak{t})} \varepsilon(w) e^{w\lambda}}{\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Here  $\varepsilon(w) = \det(w)$ , and  $W(\mathfrak{k}, \mathfrak{t})$  is the Weyl group of  $(\mathfrak{k}, \mathfrak{t})$ . The character  $\chi_{\Delta_{d_p}}$  of the representation

$$K \xrightarrow{\widetilde{\text{Ad}}} \text{Spin}(\mathfrak{p}) \rightarrow \text{GL}(\Delta_{d_p}), \quad (6.16)$$

on the other hand, is given by (Parthasarathy [65], Remark 2.2)

$$\chi_{\Delta_{d_p}}|_{T^{\text{reg}}} := (\chi_{\Delta_{d_p}^+} - \chi_{\Delta_{d_p}^-})|_{T^{\text{reg}}} = \prod_{\alpha \in R_n^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

It follows from this formula that for all  $t \in T^{\text{reg}}$ ,

$$\chi_{\Delta_{d_p}^*}(t) = \overline{\chi_{\Delta_{d_p}}(t^{-1})} = \chi_{\Delta_{d_p}}(t),$$

and hence

$$\begin{aligned} (\Theta_\lambda \chi_{\Delta_{d_p}^*})|_{T^{\text{reg}}} &= (-1)^{\frac{\dim G/K}{2}} \frac{\sum_{w \in W(\mathfrak{k}, \mathfrak{t})} \varepsilon(w) e^{w\lambda}}{\prod_{\alpha \in R_c^+} (e^{\alpha/2} - e^{-\alpha/2})} \\ &= (-1)^{\frac{\dim G/K}{2}} \chi_{\lambda - \rho_c}, \end{aligned}$$

by Weyl's character formula. Here  $\chi_{\lambda - \rho_c}$  is the character of the irreducible representation of  $K$  with highest weight  $\lambda - \rho_c$ .

Therefore, by (6.15),

$$\begin{aligned} R_G^{\mathcal{H}}(\text{D-Ind}_K^G[V_\mu]) &= [\Delta_{d_p}^* \otimes \mathcal{H}|_K : V_\mu] \\ &= (-1)^{\frac{\dim G/K}{2}} [V_{\lambda - \rho_c} : V_\mu] \\ &= \begin{cases} (-1)^{\frac{\dim G/K}{2}} & \text{if } \mu = \lambda - \rho_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

**Remark 6.10.** Lemma 6.9 is strictly speaking not an orbit method, because the coadjoint orbit through  $\mu$  is only equal to  $G/K$  if  $K = T$ , and  $\mu$  does not lie on any root hyperplanes.

## 6.3 Quantisation commutes with reduction at discrete series representations of semisimple groups

Consider the situation of Section 6.1, with the additional assumptions and notation of Section 6.2. We will state a rigorous version of (6.5) in this setting, under the assumption that the image of  $\Phi$  lies inside the *strongly elliptic set*  $\mathfrak{g}_{\text{se}}^* \subset \mathfrak{g}^*$ . We first clarify this assumption, and then state our result for semisimple groups.



### The set $\mathfrak{g}_{\text{se}}^*$

Let us define the subset  $\mathfrak{g}_{\text{se}}^* \subset \mathfrak{g}^*$  of *strongly elliptic elements*. We always view  $\mathfrak{k}^*$  as a subspace of  $\mathfrak{g}^*$  via the linear isomorphism  $\mathfrak{k}^* \cong \mathfrak{p}^0$  (via restriction from  $\mathfrak{g}$  to  $\mathfrak{k}$ ), with  $\mathfrak{p}^0$  the annihilator of  $\mathfrak{p}$  in  $\mathfrak{g}^*$ . As before, the dual space  $\mathfrak{t}^*$  is identified with the subspace  $(\mathfrak{k}^*)^{\text{Ad}^*(T)}$  of  $\mathfrak{k}^*$ .

Let  $\mathfrak{t}_+^* \subset \mathfrak{t}^*$  be a choice of positive Weyl chamber. We denote by ‘ncw’ the set of noncompact walls:

$$\text{ncw} := \{\xi \in \mathfrak{t}^*; (\alpha, \xi) = 0 \text{ for some } \alpha \in R_n\}, \quad (6.17)$$

where as before,  $(-, -)$  is a Weyl group invariant inner product on  $\mathfrak{t}_{\mathbb{C}}^*$ . We then define

$$\mathfrak{g}_{\text{se}}^* := \text{Ad}^*(G)(\mathfrak{t}_+^* \setminus \text{ncw}). \quad (6.18)$$

Equivalently,  $\mathfrak{g}_{\text{se}}^*$  is the set of all elements of  $\mathfrak{g}^*$  with compact stabilisers under the coadjoint action, and also the interior of the elliptic set  $\mathfrak{g}_{\text{ell}}^* := \text{Ad}(G)\mathfrak{k}^*$ . We will also use the notation

$$\mathfrak{k}_{\text{se}}^* := \text{Ad}^*(K)(\mathfrak{t}_+^* \setminus \text{ncw}). \quad (6.19)$$

Note that  $\mathfrak{k}_{\text{se}}^* \subset \mathfrak{k}^*$  is an open dense subset, and that  $\mathfrak{g}_{\text{se}}^* = \text{Ad}^*(G)\mathfrak{k}_{\text{se}}^*$ . The set  $\mathfrak{g}_{\text{se}}^*$  is generally not dense in  $\mathfrak{g}^*$ .

The reason for our assumption that the momentum map takes values in  $\mathfrak{g}_{\text{se}}^*$  is that we are looking at multiplicities of discrete series representations. These can be seen as ‘quantisations’ of certain coadjoint orbits that lie inside  $\mathfrak{g}_{\text{se}}^*$  (see Schmid [69], Parthasarathy [65] and also Paradan [64]). In general, the ‘quantisation commutes with reduction’ principle implies that the quantisation of a Hamiltonian action decomposes into irreducible representations associated to coadjoint orbits that lie in the image of the momentum map. Hence if we suppose that this image lies inside  $\mathfrak{g}_{\text{se}}^*$ , we expect the quantisation of the action to decompose into discrete series representations. In [89], Proposition 2.6, Weinstein proves that  $\mathfrak{g}_{\text{se}}^*$  is nonempty if and only if  $\text{rank } G = \text{rank } K$ , which is Harish-Chandra’s criterion for the existence of discrete series representations of  $G$ .

The most direct application of the assumption that the image of the momentum map lies in  $\mathfrak{g}_{\text{se}}^*$  is the following lemma, which we will use several times.

**Lemma 6.11.** *Let  $\xi \in \mathfrak{g}_{\text{se}}^*$ . Then  $\mathfrak{g}_{\xi} \cap \mathfrak{p} = \{0\}$ .*

*Proof.* Let  $X \in \mathfrak{g}_{\xi} \cap \mathfrak{p}$  be given. We consider the one-parameter subgroup  $\exp(\mathbb{R}X)$  of  $G$ . Because  $\xi \in \mathfrak{g}_{\text{se}}^*$ , the stabiliser  $G_{\xi}$  is compact. Because  $\exp(\mathbb{R}X)$  is contained in  $G_{\xi}$ , it is therefore either the image of a closed curve, or dense in a subtorus of  $G_{\xi}$ . In both cases, its closure is compact.

On the other hand, the map  $\exp : \mathfrak{p} \rightarrow G$  is an embedding (see e.g. [46], Theorem 6.31c). Hence, if  $X \neq 0$ , then  $\exp(\mathbb{R}X)$  is a closed subset of  $G$ , diffeomorphic to  $\mathbb{R}$ . Because the closure of  $\exp(\mathbb{R}X)$  is compact by the preceding argument, we conclude that  $X = 0$ .  $\square$

Now suppose that  $\Phi(M) \subset \mathfrak{g}_{\text{se}}^*$ . Then the assumption that the action of  $G$  on  $M$  is proper is actually unnecessary:

**Lemma 6.12.** *If  $\Phi(M) \subset \mathfrak{g}_{\text{se}}^*$ , then the action of  $G$  on  $M$  is automatically proper.*

*Proof.* In [89], Corollary 2.13, it is shown that the coadjoint action of  $G$  on  $\mathfrak{g}_{\text{se}}^*$  is proper. This is a slightly stronger property than the fact that elements of  $\mathfrak{g}_{\text{se}}^*$  have compact stabilisers, and it implies properness of the action of  $G$  on  $M$ .

Indeed, let a compact subset  $C \subset M$  be given. It then follows from continuity and equivariance of  $\Phi$ , and from properness of the action of  $G$  on  $\mathfrak{g}_{\text{se}}^*$  that the closed set

$$\begin{aligned} G_C &:= \{g \in G; gC \cap C \neq \emptyset\} \\ &\subset \{g \in G; g\Phi(C) \cap \Phi(C) \neq \emptyset\} \end{aligned}$$

is compact, i.e. the action of  $G$  on  $M$  is proper.  $\square$

## The result

Compactness of  $M/G$  is enough to guarantee compactness of the reduced spaces  $M_\xi = \Phi^{-1}(\xi)/G_\xi \cong \Phi^{-1}(G \cdot \xi)/G$ , but it can even be shown that in this setting,  $\Phi$  is a proper map. This gives another reason why the reduced spaces are compact.

We can finally state our result. Let  $\mathcal{H}$  be an irreducible discrete series representation. Let  $\lambda \in \mathfrak{t}^*$  be its Harish-Chandra parameter such that  $(\alpha, \lambda) > 0$  for all  $\alpha \in R_c^+$ . As before, we will write  $(M_\lambda, \omega_\lambda) := (M_{-i\lambda}, \omega_{-i\lambda})$  for the symplectic reduction of  $(M, \omega)$  at  $-i\lambda \in \mathfrak{t}_+^* \setminus \text{new} \subset \mathfrak{g}_{\text{se}}^*$ . Then our generalisation of Theorem 3.38 is:

**Theorem 6.13** (Quantisation commutes with reduction at discrete series representations). *Consider the situation of Conjecture 6.4, with the difference that  $(M, \omega)$  is now supposed to have a  $G$ -equivariant  $\text{Spin}^c$ -prequantisation  $(L^\omega, (-, -)_{L^\omega}, \nabla)$  instead of a normal one. Suppose that the additional assumptions of this section hold, and that the action of  $G$  on  $M$  has abelian stabilisers. If  $-i\lambda$  is a regular value of  $\Phi$ , then*

$$R_G^{\mathcal{H}}(Q_{VI}(M, \omega)) := R_G^{\mathcal{H}}(\mu_M^G[\mathcal{D}_M^{L^{2\omega}}]) = (-1)^{\frac{\dim G/K}{2}} Q_N(M_\lambda, \omega_\lambda).$$

*If  $-i\lambda$  does not lie in the image of  $\Phi$ , then the integer on the left hand side equals zero.*

We will prove this theorem in Part IV, via a reduction to the compact case.

As in Theorem 6.5, we use the compact version of quantisation to define the quantisation  $Q_N(M_\lambda, \omega_\lambda)$  of the symplectic reduction, since this version is well-defined in the orbifold case.

If  $G = K$ , then the irreducible discrete series representation  $\mathcal{H}$  is the irreducible representation  $V_{\lambda - \rho_c}$  of  $K$  with highest weight  $\lambda - \rho_c$  (see [69], corollary on page 105). Hence  $R_G^{\mathcal{H}}$  amounts to taking the multiplicity of  $V_{\lambda - \rho_c}$ , as remarked after the definition of  $R_G^{\mathcal{H}}$ . The assumption that  $M/G$  is compact is now equivalent to compactness of  $M$  itself. Therefore Theorem 6.13 indeed reduces to Theorem 3.38 in this case. As mentioned before, our proof of Theorem 6.13 is based on this statement for the compact case, so that we cannot view Theorem 3.38 as a corollary to Theorem 6.13.

To obtain results about discrete series representations, we would like to apply Theorem 6.13 to cases where  $M$  is a coadjoint orbit of some semisimple group, such that the quantisation of this orbit in the sense of Definition 6.1 is the  $K$ -theory class of a discrete series representation of this group. The condition that  $M/G$  is compact rules out any interesting applications in this direction, however. If we could generalise Theorem 6.13 to a similar statement where the

assumption that  $M/G$  is compact is replaced by the assumption that the momentum map is proper, then we might be able to deduce interesting corollaries in representation theory.

One such application could be analogous to unpublished work of Duflo and Vargas about restricting discrete series representations to semisimple subgroups. In this case, the assumption that the momentum map is proper corresponds to their assumption that the restriction map from some coadjoint orbit to the dual of the Lie algebra of such a subgroup is proper.

An interesting refinement of a special case of Duflo and Vargas's work was given by Paradan [64], who gives a multiplicity formula for the decomposition of the restriction of a discrete series representation of  $G$  to  $K$ , in terms of symplectic reductions of the coadjoint orbit corresponding to this discrete series representation.

## **Part II**

### **Naturality of the assembly map**

The two main results in this thesis are Theorems 6.5 and 6.13. We will prove these results by deducing them from the compact case, Theorems 3.34 and 3.38. This deduction is based on the results in this part, which express ‘naturality of the assembly map’. For discrete groups, this naturality is proved in Valette’s part of [61]. The proof in [61] is split into two parts: the ‘epimorphism case’ and the ‘monomorphism case’.

We first give a generalisation of Valette’s epimorphism case to possibly non-discrete groups (Theorem 7.1). The proof of this theorem is a straightforward generalisation of Valette’s.

Then, we give an explicit description of the epimorphism case for  $K$ -homology classes of equivariant elliptic differential operators. This is Corollary 8.11, which is the key result in our proof of Theorem 6.5.

Finally, we generalise a very special case of the monomorphism case to inclusions of maximal compact subgroups into semisimple Lie groups. This is Theorem 9.1, which is the central step in the ‘quantisation commutes with *induction*’ result, Theorem 14.5, in Part IV. The latter result in turn is the key to the deduction of Theorem 6.13 from Theorem 3.38.

In Parts III and IV, we show that the ‘naturality of the assembly map’ results in this part are ‘well-behaved’ with respect to the  $K$ -homology classes of the Dirac operators we use to define quantisation. These facts, together with Theorems 3.34 and 3.38, will imply Theorems 6.5 and 6.13.

This part contains almost all of the noncommutative geometry in this thesis. In Parts III and IV, we will almost only use differential and symplectic geometry (the most notable exception is Chapter 11). Readers who are less familiar with noncommutative geometry than with the other subjects of this thesis should feel free to skip the proofs in this part, and only read the main results, Theorems 7.1 and 9.1, before going on to Part III.

# Chapter 7

## The epimorphism case

Theorem 6.5 is partly a consequence of naturality of the assembly map. For discrete groups, this naturality is explained in detail by Valette in [61]. In this chapter, we generalise the ‘epimorphism part’ of Valette’s theorem to possibly non-discrete groups. This generalisation is basically a straightforward exercise in replacing sums by integrals and finite sets by compact ones. Where Valette uses the facts that finite sums of bounded operators on Hilbert spaces are bounded operators, and that finite sums of compact operators on Hilbert  $C^*$ -modules are again compact, we use the lemmas in Section 7.1. These lemmas, together with Lemma 7.18 and the final part of the proof of Theorem 7.1 are our own input, the rest of this chapter consists of slight generalisations of arguments from [61].

Throughout this chapter,  $G$  is a locally compact unimodular group, equipped with a Haar measure  $dg$ , acting properly on a locally compact Hausdorff space  $X$ . We consider a closed normal subgroup  $N$  of  $G$ , and a left-invariant Haar measure  $dn$  on  $N$ . We suppose that  $X/G$  is compact.

In Section 7.4, we will also need the assumption that either  $X/N$  or  $N$  is compact. This assumption may not be necessary, but we need it for our arguments. We will apply the results in this chapter to the case where  $N$  is compact in Section 9.1, and to the case where  $X/N$  is compact in Section 10.1.

The version of naturality of the assembly map that we will need is the following.

**Theorem 7.1.** *The Valette homomorphism  $V_N$ , defined in Section 7.4, makes the following diagram commutative:*

$$\begin{array}{ccc} K_0^G(X) & \xrightarrow{\mu_X^G} & K_0(C^*(G)) \\ V_N \downarrow & & \downarrow R_N^0 \\ K_0^{G/N}(X/N) & \xrightarrow{\mu_{X/N}^{G/N}} & K_0(C^*(G/N)). \end{array}$$

Here  $\mu_X^G$  and  $\mu_{X/N}^{G/N}$  are analytic assembly maps as explained in Section 5.2, and the map

$$R_N^0 = (\int_N)_* : K_0(C^*(G)) \rightarrow K_0(C^*(G/N)) \quad (7.1)$$

is functorially induced by the map  $\int_N : C^*(G) \rightarrow C^*(G/N)$  given on  $f \in C_c(G)$  by [24]

$$\int_N(f) : Ng \mapsto \int_N f(ng) dn. \quad (7.2)$$

In Chapter 8, we describe the image of a  $K$ -homology class defined by an elliptic differential operator under the homomorphism  $V_N$  (see Corollary 8.11). This description will allow us to prove Theorem 6.5 in Part III.

A version of naturality of the assembly map for locally compact groups can also be distilled from [14].

Sections 7.1–7.3 consist of preparations for the definition of the homomorphism  $V_N$  in Section 7.4, and for the proof of Theorem 7.1 in Section 7.5.

## 7.1 Integrals of families of operators

In this chapter, there are several occasions where we consider integrals of families of operators. The following facts will be used in those situations.

### Adjointable operators and integrals

**Lemma 7.2.** *Let  $(M, \mu)$  be a compact Borel space with finite measure, let  $\mathcal{E}$  be a Hilbert  $A$ -module, and let*

$$\varphi : M \rightarrow \mathcal{B}(\mathcal{E})$$

*be a continuous map. Then the integral*

$$\int_M \varphi(m) d\mu(m)$$

*defines an adjointable operator on  $\mathcal{E}$ , determined by*

$$\left( \xi, \int_M \varphi(m) d\mu(m) \eta \right)_{\mathcal{E}} = \int_M (\xi, \varphi(m) \eta)_{\mathcal{E}} d\mu(m) \in A, \quad (7.3)$$

*for all  $\xi, \eta \in \mathcal{E}$ .*

*Proof.* The integral on the right hand side of (7.3) converges, because  $\mu(M)$  is finite, and because the map  $m \mapsto (\varphi(m)\xi, \eta)_{\mathcal{E}}$  is continuous on the compact space  $M$ , and hence bounded. It follows directly from the definition (7.3) of the operator  $\int_M \varphi(m) d\mu(m)$  that it has an adjoint, given by

$$\left( \int_M \varphi(m) d\mu(m) \right)^* = \int_M \varphi(m)^* d\mu(m).$$

□

We will often use the fact that ‘adjointable operators commute with integrals’, in the following sense:

**Lemma 7.3.** *Let  $(M, \mu)$  be a measure space, let  $\mathcal{E}$  be a Hilbert  $A$ -module, and let*

$$\varphi : M \rightarrow \mathcal{B}(\mathcal{E})$$

*be a measurable function. That is to say, the integral  $\int_M \varphi(m) d\mu(m)$  is a well-defined adjointable operator on  $\mathcal{E}$ , determined by (7.3).*

Let  $\mathcal{E}'$  be another Hilbert  $A$ -module, and let  $T : \mathcal{E} \rightarrow \mathcal{E}'$  be an adjointable operator. Then

$$\int_M T \circ \varphi(m) d\mu(m) = T \circ \int_M \varphi(m) d\mu(m).$$

*Proof.* The statement follows directly from (7.3).  $\square$

## Compact operators and integrals

In the proof of Lemma 7.12 we will use the fact that in some cases, ‘the integral over a compact set of a family of compact operators is compact’. To be more precise:

**Lemma 7.4.** *Let  $(M, \mu)$  be a compact Borel space with finite measure. Let  $\mathcal{E}$  be a Hilbert  $C^*$ -module, and let  $\varphi : M \rightarrow \mathcal{K}(\mathcal{E})$  be a continuous compact operator-valued map. Suppose that  $\varphi$  is ‘uniformly compact’, in the sense that there exists a sequence  $(\varphi_j)_{j=1}^\infty : M \rightarrow \mathcal{F}(\mathcal{E})$  such that*

$$\|\varphi_j - \varphi\|_\infty := \sup_{m \in M} \|\varphi_j(m) - \varphi(m)\|_{\mathcal{B}(\mathcal{E})}$$

*tends to zero as  $j \rightarrow \infty$ . Suppose furthermore that for every  $j \in \mathbb{N}$ , there is a sequence  $(\varphi_j^k)_{k=1}^\infty : M \rightarrow \mathcal{F}(\mathcal{E})$  of simple functions (i.e. measurable functions having finitely many values), such that for all  $\varepsilon > 0$  there is an  $n \in \mathbb{N}$  such that for all  $j, k \geq n$ ,  $\|\varphi_j^k - \varphi_j\| < \varepsilon$ . Then the integral*

$$\int_M \varphi(m) d\mu(m)$$

*defines a compact operator on  $\mathcal{E}$ .*

*Proof.* For all  $j, k \in \mathbb{N}$ , the integral  $\int_M \varphi_j^k(m) d\mu(m)$  is a finite sum of finite rank operators, and hence a finite rank operator itself. And because  $\|\varphi_j^j - \varphi\|_\infty \rightarrow 0$  as  $j$  tends to  $\infty$ , we have

$$\int_M \varphi_j^j(m) d\mu(m) \rightarrow \int_M \varphi(m) d\mu(m)$$

in  $\mathcal{B}(\mathcal{E})$ . Hence  $\int_M \varphi(m) d\mu(m)$  is a compact operator.  $\square$

In the following situation, the assumptions of Lemma 7.4 are met:

**Lemma 7.5.** *Let  $\mathcal{E}$  be a Hilbert  $C^*$ -module, and let  $(M, \mu)$  be a compact Borel space with finite measure. Suppose  $M$  is metrisable. Let  $\alpha, \beta : M \rightarrow \mathcal{B}(\mathcal{E})$  be continuous, and let  $T \in \mathcal{K}(\mathcal{E})$  be a compact operator. Define the map  $\varphi : M \rightarrow \mathcal{K}(\mathcal{E})$  by  $\varphi(m) = \alpha(m)T\beta(m)$ . This map satisfies the assumptions made in Lemma 7.4.*

*Proof.* Choose a sequence  $(T_j)_{j=1}^\infty$  in  $\mathcal{F}(\mathcal{E})$  that converges to  $T$ . For  $m \in M$ , set

$$\varphi_j(m) = \alpha(m)T_j\beta(m)$$

Then

$$\|\varphi_j - \varphi\|_\infty \leq \|\alpha\|_\infty \|T_j - T\|_{\mathcal{B}(\mathcal{E})} \|\beta\|_\infty \rightarrow 0$$



as  $j \rightarrow \infty$ . Note that  $\alpha$  and  $\beta$  are continuous functions on a compact space, so their sup-norms are finite.

Choose sequences of simple functions  $\alpha^k, \beta^k : M \rightarrow \mathcal{B}(\mathcal{E})$  such that  $\|\alpha^k - \alpha\|_\infty \rightarrow 0$  and  $\|\beta^k - \beta\|_\infty \rightarrow 0$  as  $j$  goes to  $\infty$  (see Lemma 7.6 below). For all  $j, k \in \mathbb{N}$ , set

$$\phi_j^k(m) := \alpha^k(m)T_j\beta^k(m),$$

for  $m \in M$ . Note that

$$\begin{aligned} \|\phi_j^k - \phi_j\|_\infty &= \sup_{m \in M} \|\alpha^k(m)T_j\beta^k(m) - \alpha(m)T_j\beta(m)\| \\ &\leq \sup_{m \in M} \left( \|\alpha^k(m)T_j\beta^k(m) - \alpha^k(m)T_j\beta(m)\| \right. \\ &\quad \left. + \|\alpha^k(m)T_j\beta(m) - \alpha(m)T_j\beta(m)\| \right) \\ &\leq \|\alpha^k\|_\infty \|T_j\| \|\beta^k - \beta\|_\infty + \|\alpha^k - \alpha\|_\infty \|T_j\| \|\beta\|_\infty. \end{aligned}$$

The sequences  $k \mapsto \|\alpha^k\|_\infty$  and  $j \mapsto \|T_j\|$  are bounded, since  $\alpha^k \rightarrow \alpha$  and  $T_j \rightarrow T$ . Hence, because the sequences  $\|\alpha^k - \alpha\|_\infty$  and  $\|\beta^k - \beta\|_\infty$  tend to zero, we see that  $\|\phi_j^k - \phi_j\|$  can be made smaller than any  $\varepsilon > 0$  for  $k$  large enough, uniformly in  $j$ .  $\square$

**Lemma 7.6.** *Let  $(M, \mu)$  be a metrisable compact Borel space with metric  $d$ , let  $Y$  be a normed vector space, and let  $\alpha : M \rightarrow Y$  be a continuous map.*

*Then there exists a sequence of simple maps  $\alpha^k : M \rightarrow Y$  such that the sequence*

$$\|\alpha - \alpha^k\|_\infty := \sup_{m \in M} \|\alpha(m) - \alpha^k(m)\|_Y$$

*goes to zero as  $k$  goes to infinity.*

*Proof.* For every  $k \in \mathbb{N}$ , choose a finite covering  $\tilde{U}_k = \{\tilde{V}_k^1, \dots, \tilde{V}_k^{n_k}\}$  of  $M$  by balls of radius  $\frac{1}{k}$ . From each  $\tilde{U}_k$ , we construct a partition  $U_k = \{V_k^1, \dots, V_k^{n_k}\}$  of  $M$ , by setting  $V_k^1 := \tilde{V}_k^1$ , and  $V_k^j := \tilde{V}_k^j \setminus \bigcup_{i=1}^{j-1} \tilde{V}_k^i$ , for  $j = 2, \dots, n_k$ . Note that the sets  $V_k^j$  are Borel-measurable. For all  $k \in \mathbb{N}$  and  $j \in \{1, \dots, n_k\}$ , choose an element  $m_k^j \in V_k^j$ . Define the simple map  $\alpha^k : M \rightarrow Y$  by

$$\alpha^k(m) := \alpha(m_k^j) \quad \text{if } m \in V_k^j.$$

Note that, because  $\alpha$  is continuous (and uniformly continuous because  $M$  is compact), for every  $\varepsilon > 0$  there is a  $k_\varepsilon \in \mathbb{N}$  such that for all  $m, n \in M$ ,

$$d(m, n) < \frac{1}{k_\varepsilon} \quad \Rightarrow \quad \|\alpha(m) - \alpha(n)\|_Y < \varepsilon.$$

Hence for all  $\varepsilon > 0$ , all  $k > k_\varepsilon$ , and all  $m \in M$  (say  $m \in V_k^j$ ),

$$\|\alpha(m) - \alpha^k(m)\|_Y = \|\alpha(m) - \alpha(m_k^j)\|_Y < \varepsilon.$$

So  $\|\alpha - \alpha^k\|_\infty$  indeed goes to zero.  $\square$

## 7.2 Extension of operators to Hilbert $C^*$ -modules

From now on, let  $(\mathcal{H}, F, \pi)$  be a  $G$ -equivariant  $K$ -homology cycle over  $X$ . In the definition of the assembly map, a Hilbert  $C^*(G)$ -module  $\mathcal{E}$  is constructed from the Hilbert space  $\mathcal{H}$ , namely as the closure of the space  $\mathcal{H}_c = \pi(C_c(X))\mathcal{H}$  in a certain norm (see Section 5.2). We shall prove the well-known fact that  $F$  induces an operator on  $\mathcal{E}$ , because we will also use some of the ingredients in this proof later in this chapter.

**Lemma 7.7.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be properly supported and  $G$ -equivariant. Then  $T$  preserves  $\mathcal{H}_c$ , and  $T|_{\mathcal{H}_c}$  extends continuously to an adjointable operator  $T_{\mathcal{E}}$  on  $\mathcal{E}$ .*

In the proof of Lemma 7.7, and also later, we will use:

**Lemma 7.8.** *There is a nonnegative function  $c \in C_c(X)$  such that for all  $x \in X$ ,*

$$\int_G c(gx) dg = 1.$$

*Proof.* Because the quotient  $X/G$  is compact, there is a nonnegative function  $h \in C_c(X)$  such that for all  $x \in X$ , the orbit  $Gx$  intersects the interior of the support of  $h$ . Therefore,

$$\int_G h(gx) dx > 0$$

for all  $x \in X$ . Let  $c \in C_c(X)$  be the function

$$c(x) := \frac{h(x)}{\int_G h(gx) dg}.$$

By right invariance of  $dg$ , this function has the desired property.  $\square$

**Corollary 7.9.** *Let  $h_N \in C_c(X/N)$ . Then there is a function  $h \in C_c(X)$  such that for all  $x \in X$ ,*

$$\int_N h(nx) dn = h_N(Nx). \quad (7.4)$$

*Proof.* If  $X/N$  is compact, choose

$$h(x) := c(x)h_N(Nx),$$

where  $c$  is the function from Lemma 7.8 (with  $G$  replaced by  $N$ ). Otherwise set  $Y := p^{-1}(\text{supp } h_N)$ , with  $p : X \rightarrow X/N$  the quotient map. The preceding argument yields a function  $h \in C_c(Y)$  such that for all  $y \in Y$ ,

$$\int_N h(ny) dn = h_N(Ny).$$

Since  $\partial Y = p^{-1}(\partial \text{supp } h_N)$ , we have  $h|_{\partial Y} = 0$ . Hence  $h$  can be extended by zero outside  $Y$  to a continuous function on  $X$ . This extension satisfies (7.4).  $\square$

### An auxiliary map $S$

Let  $L^2(G, \mathcal{H})$  be the Hilbert space of functions  $\varphi : G \rightarrow \mathcal{H}$  whose norm-squared function  $g \mapsto (\varphi(g), \varphi(g))_{\mathcal{H}}$  is integrable over  $G$ . Let  $c \in C_c(X)$  be the function from Lemma 7.8, and let  $f := \sqrt{c}$ . Just as Valette does in [61], we define the linear map

$$S : \mathcal{H} \rightarrow L^2(G, \mathcal{H})$$

by

$$S\xi(g) = \pi(f)g \cdot \xi.$$

**Lemma 7.10.** *The map  $S$  is an isometry, intertwines the representation of  $G$  in  $\mathcal{H}$  and the right regular representation of  $G$  in  $L^2(G, \mathcal{H})$ , and it maps  $\mathcal{H}_c$  into the space  $L_c^2(G, \mathcal{H})$  of compactly supported  $L^2$ -functions from  $G$  to  $\mathcal{H}$ .*

*Proof.* The facts that  $G$  acts unitarily on  $\mathcal{H}$ ,  $\pi$  is a  $*$ -homomorphism and a nondegenerate representation, together with Lemma 7.3 and the definition of  $f$ , imply that  $S$  is an isometry. So in particular, the image of  $S$  lies inside  $L^2(G, \mathcal{H})$ . Furthermore, it follows from the definitions that  $S$  intertwines the representation of  $G$  in  $\mathcal{H}$  and the right regular representation of  $G$  in  $L^2(G, \mathcal{H})$ .

By equivariance of  $\pi$ , we have for all  $h \in C_c(X)$ , all  $\xi \in \mathcal{H}$  and all  $g \in G$ ,

$$S(\pi(h)\xi)(g) = \pi(f)g\pi(h)\xi = \pi(fgh)g \cdot \xi.$$

Since the action of  $G$  on  $X$  is proper, the latter expression is a compactly supported function of  $g$ . In other words, the image of the space  $\mathcal{H}_c$  under the map  $S$  is contained in the space  $L_c^2(G, \mathcal{H})$ .  $\square$

The spaces  $\mathcal{H}_c$  and  $L_c^2(G, \mathcal{H})$  carry  $C_c(G) \subset C^*(G)$ -valued inner products given by

$$(\xi, \eta)_{C^*(G)}(g) = (\xi, g \cdot \eta)_{\mathcal{H}}, \quad (7.5)$$

for  $\xi, \eta \in \mathcal{H}_c$  and  $g \in G$ , and

$$(\varphi, \psi)_{C^*(G)}(g) = (\varphi, \rho^G(g)\psi)_{L^2(G, \mathcal{H})}, \quad (7.6)$$

for  $\varphi, \psi \in L_c^2(G, \mathcal{H})$  and  $g \in G$ . Here  $\rho^G$  denotes the right regular representation of  $G$  in  $L^2(G, \mathcal{H})$ :  $(\rho^G(g)\psi)(g') = \psi(g'g)$ . With respect to these inner products, the adjoint of the restriction  $S : \mathcal{H}_c \rightarrow L_c^2(G, \mathcal{H})$  is the map

$$S^* : L_c^2(G, \mathcal{H}) \rightarrow \mathcal{H}_c$$

given by

$$S^*\varphi = \int_G g^{-1} \pi(f) \varphi(g) dg. \quad (7.7)$$

This follows from a computation involving an application of Lemma 7.3.

Another important property of the maps  $S$  and  $S^*$  is that the composition  $S^*S$  is the identity on  $\mathcal{H}_c$ , by definition of  $f$  and by Lemma 7.3.

*Proof of Lemma 7.7.* Because  $T$  is properly supported, it preserves  $\mathcal{H}_c$ . Via the map  $S$ , the restriction of  $T$  to  $\mathcal{H}_c$  induces the operator  $STS^*$  on  $L_c^2(G, \mathcal{H}) \cong L_c^2(G) \hat{\otimes} \mathcal{H}$ , which is a dense subspace of the Hilbert  $C^*(G)$ -module  $C^*(G) \otimes \mathcal{H}$ . This embedding of  $L_c^2(G, \mathcal{H})$  into  $C^*(G) \otimes \mathcal{H}$  is isometric with respect to the  $C^*(G)$ -valued inner product (7.6) on  $L_c^2(G, \mathcal{H})$  and the  $C^*(G)$ -valued inner product on  $C^*(G) \otimes \mathcal{H}$  given by

$$(\alpha \otimes \xi, \beta \otimes \eta)_{C^*(G) \otimes \mathcal{H}} = (\xi, \eta)_{\mathcal{H}} \alpha * \beta^*,$$

for  $\alpha, \beta \in C^*(G)$  and  $\xi, \eta \in \mathcal{H}$ . We will show that the operator  $STS^*$  defines an adjointable operator on  $C^*(G) \otimes \mathcal{H}$  with respect to this inner product. We then conclude that  $T = S^*STS^*S$  is adjointable as well.

To see that  $STS^*$  defines an adjointable operator on  $C^*(G) \otimes \mathcal{H}$ , let  $\varphi \in L_c^2(G, \mathcal{H})$  be given. Then for all  $g \in G$ , one computes

$$STS^*\varphi(g) = \int_G \pi(f)T\pi(g'f)g'\varphi(g'^{-1}g)dg'.$$

Identifying  $L_c^2(G, \mathcal{H})$  with  $L_c^2(G) \hat{\otimes} \mathcal{H}$ , we see that for all  $\chi \in L_c^2(G)$  and  $\xi \in \mathcal{H}$ ,

$$STS^*(\chi \otimes \xi) = \int_G \chi(g'^{-1}g) \pi(f)T\pi(g'f)g'\xi dg'.$$

In other words,

$$STS^* = \int_G \lambda^G(g') \otimes (\pi(f)T\pi(g'f)g') dg', \quad (7.8)$$

where  $\lambda^G$  denotes the left regular representation of  $G$  in  $L^2(G)$ .

The integrand in (7.8) is compactly supported, since by equivariance of  $\pi$  and  $T$ ,

$$\pi(f)T\pi(g'f) = \pi(f)g'T\pi(f)g'^{-1} = g'\pi(g'^{-1}f)\pi(h)T\pi(f)g'^{-1}$$

for some  $h \in C_c(X)$ , because  $T$  is properly supported. And because the action of  $G$  on  $X$  is proper, the map

$$g' \mapsto \pi(g'^{-1}f)\pi(h) = \pi((g'^{-1}f)h)$$

has compact support  $\Sigma$ . Note that, for  $\chi, \chi' \in L_c^2(G)$  and  $\xi, \xi' \in \mathcal{H}$ , the  $C_c(G)$ -valued inner product (7.6) is given by

$$(\chi \otimes \xi, \chi' \otimes \xi')_{C^*(G)}(g) = (\chi, \rho^G(g)\chi')_{L^2(G)}(\xi, \xi')_{\mathcal{H}},$$

for  $g \in G$ . Since by Lemma 7.2, the operators  $\int_{\Sigma} \pi(f)T\pi(g'f)g'dg'$  on  $\mathcal{H}$  and  $\int_{\Sigma} \lambda^G(g')dg'$  on  $L^2(G)$  are adjointable, and since the left and right regular representations of  $G$  in  $L^2(G)$  commute, the operator  $S^*TS^*$  is adjointable.  $\square$

## 7.3 The averaging process

In the proof that the homomorphism  $V_N$  is well-defined, we will use a certain averaging process.

## Averaging

As before, let  $(\mathcal{H}, F, \pi)$  be an equivariant  $K$ -homology cycle over  $X$ .

**Lemma 7.11.** *For  $T \in \mathcal{B}(\mathcal{H})$  and  $f \in C_c(X)$ , set*

$$A_f^G(T) := \int_G g \pi(f) T \pi(f) g^{-1} dg.$$

1.  $A_f^G(T)$  is a well-defined bounded operator on  $\mathcal{H}$ ;
2.  $A_f^G(T)$  is properly supported;
3.  $A_f^G(T)$  is  $G$ -equivariant.

*Proof.* 1. Suppose  $T$  is self-adjoint. (Otherwise apply the following argument to the real and imaginary parts of  $T$ .) Then for all  $g \in G$ , we have the inequalities in  $\mathcal{B}(\mathcal{H})$ :

$$-g \pi(f^2) g^{-1} \|T\|_{\mathcal{B}(\mathcal{H})} 1_{\mathcal{H}} \leq g \pi(f) T \pi(f) g^{-1} \leq g \pi(f^2) g^{-1} \|T\|_{\mathcal{B}(\mathcal{H})} 1_{\mathcal{H}}.$$

Therefore,

$$-\int_G g \pi(f^2) g^{-1} \|T\|_{\mathcal{B}(\mathcal{H})} 1_{\mathcal{H}} dg \leq A_f^G(T) \leq \int_G g \pi(f^2) g^{-1} \|T\|_{\mathcal{B}(\mathcal{H})} 1_{\mathcal{H}} dg.$$

And hence, by equivariance property (4.14) of  $\pi$ ,

$$\begin{aligned} \|A_f^G(T)\| &\leq \left\| \int_G g \pi(f^2) g^{-1} dg \right\| \|T\| \\ &= \left\| \int_G \pi(g \cdot f^2) dg \right\| \|T\| \\ &= \left\| \pi \left( \int_G g \cdot f^2 dg \right) \right\| \|T\| \\ &\leq \left\| \int_G g \cdot f^2 dg \right\|_{\infty} \|T\|, \end{aligned}$$

where we have used the fact that the function

$$x \mapsto \int_G f^2(gx) dg$$

is in  $C(X)^G \cong C(X/G)$ , and hence bounded, by compactness of  $X/G$ .

2. Let  $\varphi \in C_c(X)$ . Then, using equivariance of  $\pi$  in the second equality, we see that

$$\begin{aligned} A_f^G(T) \pi(\varphi) &= \int_G g \pi(f) T \pi(f) g^{-1} \pi(\varphi) dg \\ &= \int_G \pi(g \cdot f) g T \pi(f g^{-1} \cdot \varphi) g^{-1} dg. \end{aligned} \tag{7.9}$$

Let  $K \subset G$  be the compact set  $K := \{g \in G; f g^{-1} \varphi \neq 0\}$ . This set is compact because the  $G$ -action on  $X$  is proper. Choose a function  $\psi \in C_c(X)$  that equals 1 on the compact set  $\bigcup_{g \in K} g \cdot \text{supp } f$ . Then, since  $\psi g \cdot f = g \cdot f$  for all  $g \in K$ , it follows from (7.9) that

$$\pi(\psi) A_f^G(T) \pi(\varphi) = A_f^G(T) \pi(\varphi).$$

3. Equivariance of  $A_f^G(T)$  follows from left invariance of the Haar measure  $dg$ . □

### Averaging compact operators

Let  $T$  be a bounded operator on  $\mathcal{H}$ , and let  $h \in C_c(X)$  be given. Then by Lemma 7.11, the averaged operator  $A_h^G(T)$  is properly supported and  $G$ -equivariant. So by Lemma 7.7, the operator  $A_h^G(T)$  induces an adjointable operator on  $\mathcal{E}$ . We will need the following lemma to prove that the homomorphism  $V_N$  is well-defined.

**Lemma 7.12.** *If  $T$  is a compact operator, then the operator on  $\mathcal{E}$  induced by  $A_h^G(T)$  is compact as well.*

*Proof.* Let  $c \in C_c(X)$  be the function from Lemma 7.8, let  $f := \sqrt{c}$ , and let  $S$  be the operator from Lemma 7.10. Applying (7.8) to the operator  $A_h^G(T)$ , we obtain

$$\begin{aligned} SA_h^G(T)S^* &= \int_G \int_G \pi(f) (g' \pi(h) T \pi(h) g'^{-1}) \pi(g \cdot f) (\lambda_G(g) \otimes g) dg dg' \\ &= \int_G \int_G \pi(fg' \cdot h) g' T \pi(hg'^{-1}g \cdot f) (\lambda_G(g) \otimes g'^{-1}g) dg dg', \end{aligned} \quad (7.10)$$

where we have used Lemma 7.3 and equivariance of  $\pi$ .

Since the action of  $G$  on  $X$  is proper, the set  $K := \{g' \in G; fg' \cdot h \neq 0\}$  is compact. Hence the set  $L := \bigcup_{g' \in K} \{g \in G; hg'^{-1}g \cdot f \neq 0\}$  is compact as well. The support of the integrand in (7.10) is contained in  $K \times L$ , so it is compact. We see that (7.10) is the integral over a compact space of a family of compact operators. By Lemma 7.5, this family satisfies the assumptions of Lemma 7.4. The latter lemma therefore implies that  $SA_h^G(T)S^*$  defines a compact operator on  $C^*(G) \otimes \mathcal{H}$ , so that  $A_h^G(T) = S^*SA_h^G(T)S^*S$  defines a compact operator on  $\mathcal{E}$ .  $\square$

## 7.4 The homomorphism $V_N$

### Definition of $V_N$

The *Valette homomorphism*

$$V_N : K_0^G(X) \rightarrow K_0^{G/N}(X/N)$$

is given by

$$V_N[\mathcal{H}, F, \pi] = [\mathcal{H}_N, F_N, \pi_N],$$

with  $\mathcal{H}_N, F_N$  and  $\pi_N$  defined as follows.

We equip the vector space  $\mathcal{H}_c = \pi(C_c(X))\mathcal{H}$  with the sesquilinear form

$$(\xi, \eta)_N := \int_N (\xi, n \cdot \eta)_{\mathcal{H}} dn.$$

(For all  $\xi, \eta \in \mathcal{H}_c$ , the integrand is compactly supported.) This form is positive semidefinite:

**Lemma 7.13.** *For all  $\xi \in \mathcal{H}_c$ , one has*

$$(\xi, \xi)_N \geq 0. \quad (7.11)$$

*Proof.* We will prove that the compactly supported function

$$(\xi, \xi)_{C^*(N)} : n \mapsto (\xi, n \cdot \xi)_{\mathcal{H}}$$

on  $N$ , defines a positive element of  $C^*(N)$ . We then note that any homomorphism of  $C^*$ -algebras preserves positivity. Hence, applying the trivial representation to  $(\xi, \xi)_{C^*(N)}$ , we see that

$$\int_N (\xi, \xi)_{C^*(N)}(n) dn = (\xi, \xi)_N \geq 0.$$

To show that  $(\xi, \xi)_{C^*(N)}$  is a positive element of  $C^*(N)$ , we will use a map very similar to the map  $S$  of Lemma 7.10. Since  $X/N$  is not necessarily compact, the map  $S$  may not be well-defined if we replace  $G$  by  $N$ . However, write  $\xi = \pi(h)\eta$ , for some  $h \in C_c(X)$  and  $\eta \in \mathcal{H}$ . Then

$$Y := N \cdot \text{supp } h \subset X,$$

is a proper  $N$ -space, such that  $Y/N$  is compact. Therefore, by Lemma 7.8, there is a function  $f \in C_c(Y)$  such that for all  $y \in Y$ ,

$$\int_N f(n \cdot y)^2 dn = 1.$$

We define the map

$$S_\xi : \mathcal{H} \rightarrow L^2(N, \mathcal{H})$$

by

$$S_\xi(\zeta)(n) = \pi(f)n \cdot \zeta.$$

This map has similar properties to the properties of the map  $S$  given in Lemma 7.10. The adjoint of the map  $S_\xi$  with respect to the  $C^*(N)$ -valued inner products analogous to (7.5) and (7.6) is given by (7.7), with  $G$  replaced by  $N$ .

The main difference between  $S$  and  $S_\xi$  is the fact that  $S_\xi^* S_\xi$  is not the identity on  $\mathcal{H}_c$  in general. However, we do have

$$\begin{aligned} S_\xi^* S_\xi(\xi) &= \int_N n^{-1} \pi(f) \pi(f) n \pi(h) \eta dn \\ &= \int_N \pi(n \cdot f^2) dn \pi(h) \eta \\ &= \pi(h) \eta \\ &= \xi, \end{aligned}$$

since the function  $\int_N n \cdot f^2 dn$  equals 1 on the support of  $h$ . Therefore, we see that

$$(\xi, \xi)_{C^*(N)} = (\xi, S_\xi^* S_\xi \xi)_{C^*(N)} = (S_\xi \xi, S_\xi \xi)_{C^*(N)}.$$

We will shortly demonstrate that for all  $\varphi \in L_c^2(N, \mathcal{H})$ , the function  $(\varphi, \varphi)_{C^*(N)}$  is a positive element of  $C^*(N)$ . Then taking  $\varphi = S_\xi$  shows that  $(\xi, \xi)_{C^*(N)}$  is positive in  $C^*(N)$ .

Let  $\varphi \in L_c^2(N, \mathcal{H})$ , and choose a Hilbert basis  $(e_i)_{i \in I}$  of  $\mathcal{H}$ . Write  $\varphi(n) = \sum_{i \in I} \varphi_i(n) e_i$ , with  $\varphi_i \in L_c^2(N)$  for all  $i \in I$ . Then

$$\begin{aligned} (\varphi, \varphi)_{C^*(N)}(n) &= \int_N \sum_{i \in I} \bar{\varphi}_i(n') \varphi_i(n'n) dn' \\ &= \sum_{i \in I} \int_N \varphi_i^*(n'^{-1}) \varphi_i(n'n) dn' \\ &= \sum_{i \in I} \varphi_i^* * \varphi_i(n). \end{aligned}$$

Now note that all functions  $\varphi_i^* * \varphi_i$  are positive in  $C^*(N)$ . □

Because of this lemma, the form  $(-, -)_N$  induces an inner product on the quotient space  $\mathcal{H}_c / \ker(-, -)_N$ . We define  $\mathcal{H}_N$  to be the completion of  $\mathcal{H}_c / \ker(-, -)_N$  with respect to this inner product.

Next, let us define the operator  $F_N$ . From now on, we suppose that either  $X/N$  is compact, or  $N$  is compact.

Let  $\mathcal{E}_N$  be the Hilbert  $C^*(N)$ -module defined as the completion of  $\mathcal{H}_c$  with respect to the  $C^*(N)$ -valued inner product given by

$$(\xi, \eta)_{\mathcal{E}_N}(n) = (\xi, n \cdot \eta)_{\mathcal{H}}, \quad (7.12)$$

for  $\xi, \eta \in \mathcal{H}_c$  and  $n \in N$ .

First, suppose  $X/N$  is compact. Then, by Lemma 7.7, the operator  $F$  induces an adjointable operator  $F_{\mathcal{E}_N}$  on  $\mathcal{E}_N$ . Since adjointable operators are bounded, there is a  $c > 0$  such that for all  $\xi \in \mathcal{E}_N$ ,

$$\|F_{\mathcal{E}_N} \xi\|_{\mathcal{E}_N}^2 \leq c \|\xi\|_{\mathcal{E}_N}^2.$$

Therefore, the operator  $c1_{\mathcal{E}_N} - F_{\mathcal{E}_N}^* F_{\mathcal{E}_N}$  is a positive element of  $\mathcal{B}(\mathcal{E}_N)$ , which implies that for all  $\xi \in \mathcal{E}_N$ , the element  $((c - F_{\mathcal{E}_N}^* F_{\mathcal{E}_N})\xi, \xi)_{\mathcal{E}_N}$  of  $C^*(N)$  is positive. In other words,

$$(F_{\mathcal{E}_N} \xi, F_{\mathcal{E}_N} \xi)_{\mathcal{E}_N} \leq c(\xi, \xi)_{\mathcal{E}_N} \quad (7.13)$$

in  $C^*(N)$ . In particular, if  $\xi \in \mathcal{H}_c$ , and we apply the trivial representation, we can conclude that

$$(F\xi, F\xi)_N \leq c(\xi, \xi)_N.$$

Therefore,  $F$  extends continuously to a bounded operator  $F_N$  on  $\mathcal{H}_N$ .

If  $N$  is compact, then we have:

**Lemma 7.14.** *For all  $\xi \in \mathcal{H}_c$ ,*

$$(F\xi, F\xi)_N \leq \|F\|_{\mathcal{B}(\mathcal{H})}^2 (\xi, \xi)_N.$$

Hence also in this case, the operator  $F$  induces a bounded operator  $F_N$  on  $\mathcal{H}_N$ .



*Proof.* By equivariance of  $F$ , and by compactness of  $N$ , we have

$$\begin{aligned}
 (F\xi, F\xi)_N &= \int_N (F\xi, Fn \cdot \xi)_{\mathcal{H}} dn \\
 &= \frac{1}{\text{vol}(N)} \int_N \int_N (n'Fn'^{-1} \cdot \xi, Fn \cdot \xi)_{\mathcal{H}} dndn' \\
 &= \frac{1}{\text{vol}(N)} \int_N \int_N (Fn'^{-1} \cdot \xi, Fn'^{-1}n \cdot \xi)_{\mathcal{H}} dndn'. \tag{7.14}
 \end{aligned}$$

Applying Lemma 7.2, we obtain a bounded operator

$$\eta \mapsto \int_N n \cdot \eta \, dn$$

on  $\mathcal{H}$ , such that for all  $\eta, \eta' \in \mathcal{H}$ :

$$\left( \eta, \int_N n \cdot \eta' \, dn \right)_{\mathcal{H}} = \int_N (\eta, n \cdot \eta')_{\mathcal{H}} \, dn.$$

By Lemma 7.3 and left invariance of  $dn$ , we see that (7.14) equals

$$\begin{aligned}
 &\frac{1}{\text{vol}(N)} \left( F \left( \int_N n'^{-1} \cdot \xi \, dn' \right), F \left( \int_N n \cdot \xi \, dn \right) \right)_{\mathcal{H}} \\
 &= \frac{1}{\text{vol}(N)} \left( F \left( \int_N n \cdot \xi \, dn \right), F \left( \int_N n \cdot \xi \, dn \right) \right)_{\mathcal{H}} \\
 &\leq \frac{\|F\|_{\mathcal{B}(\mathcal{H})}^2}{\text{vol}(N)} \int_N \int_N (n \cdot \xi, n' \cdot \xi)_{\mathcal{H}} \, dndn' \\
 &\leq \frac{\|F\|_{\mathcal{B}(\mathcal{H})}^2}{\text{vol}(N)} \text{vol}(N) \max_{n \in N} \left( \int_N (n \cdot \xi, n' \cdot \xi)_{\mathcal{H}} \, dn' \right) \\
 &= \|F\|_{\mathcal{B}(\mathcal{H})}^2 \max_{n \in N} \int_N (\xi, n^{-1}n' \cdot \xi)_{\mathcal{H}} \, dn' \\
 &= \|F\|_{\mathcal{B}(\mathcal{H})}^2 (\xi, \xi)_N,
 \end{aligned}$$

by left invariance of  $dn$ . □

Finally, the representation  $\pi$  of  $C_0(X)$  in  $\mathcal{H}$  extends to the multiplier algebra  $C_b(X)$  of  $C_0(X)$  (see Example 4.11). We embed the algebra  $C_0(X/N)$  into  $C_b(X)$  via the isomorphism  $C(X/N) \cong C(X)^N$ . The operators on  $\mathcal{H}$  of the form  $\pi(f)$ , with  $f \in C(X)^N$ , are properly supported and  $N$ -equivariant. So by the argument used in the definition of  $F_N$ ,  $\pi$  induces a representation

$$\pi_N : C_0(X/N) \rightarrow \mathcal{B}(\mathcal{H}_N).$$

### $V_N$ is well-defined

Let us prove that the triple  $(\mathcal{H}_N, F_N, \pi_N)$  actually defines a class in  $K_0^{G/N}(X/N)$ . In the proof, we will use a different description of the Hilbert space  $\mathcal{H}_N$ .

Consider the Hilbert space  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C} = \mathcal{E}_N \otimes_{\int_N} \mathbb{C}$ , which is defined as the quotient of the tensor product  $\mathcal{E}_N \otimes \mathbb{C}$  by the equivalence relation

$$(\xi \cdot f) \otimes z \sim \xi \otimes \int_N(f)z,$$

for all  $\xi \in \mathcal{E}_N$ ,  $f \in C^*(N)$  and  $z \in \mathbb{C}$ . Here the map  $\int_N : C^*(N) \rightarrow \mathbb{C}$  is defined analogously to (6.2). That is, by

$$\int_N(f) := \int_N f(n) dn$$

for all  $f \in C_c(N)$ , and extended continuously to all of  $C^*(N)$ . The inner product on  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C}$  is given by

$$([\xi \otimes z], [\xi' \otimes z'])_{\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C}} := \int_N ((\xi, \xi')_{\mathcal{E}_N}) z \bar{z}'.$$

It is a straightforward matter to prove the following lemma:

**Lemma 7.15.** *The linear map  $\mathcal{H}_c \otimes \mathbb{C} \rightarrow \mathcal{H}_c$  given by  $\xi \otimes z \mapsto z\xi$  induces a unitary isomorphism  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C} \rightarrow \mathcal{H}_N$ .*

Using this description of  $\mathcal{H}_N$  we can now prove:

**Lemma 7.16.** *The triple  $(\mathcal{H}_N, F_N, \pi_N)$  defines a class in  $K_0^{G/N}(X/N)$ , with  $F_N$  properly supported.*

*Proof.* We will show that for all  $h_N \in C_0(X/N)$ , the bounded operators

$$[\pi_N(h_N), F_N] \quad \text{and} \quad \pi_N(h_N)(F_N^2 - 1)$$

on  $\mathcal{H}_N$  are compact. All other properties of  $K$ -homology cycles follow by a straightforward verification.

Let  $h_N \in C_c(X/N)$  be given. It is sufficient to prove the claim for all  $h_N$  in this dense subspace of  $C_0(X/N)$ . Let  $h \in C_c(X)$  be the function from Corollary 7.9. Then  $\int_N n \cdot h dn = p^*h_N$ , with  $p : X \rightarrow X/N$  the quotient map. We may suppose that  $h_N$  is real-valued, for otherwise we can apply the following argument to the real and imaginary parts of  $h_N$ .

We split the proof of Lemma 7.16 into two parts, by first considering the case where  $X/N$  is compact, and then proving the result for compact  $N$ .

Assume that  $X/N$  is compact. Then we have the bounded operator  $F_{\mathcal{E}_N}$  on  $\mathcal{E}_N$  induced by  $F$  as in Lemma 7.7. The isomorphism  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C} \cong \mathcal{H}_N$  from Lemma 7.15 intertwines the operator  $F_N$  on  $\mathcal{H}_N$  and the operator  $F_{\mathcal{E}_N} \otimes 1$  on  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C}$ . Indeed, for all  $\xi \in \mathcal{H}_c$  and all  $z \in \mathbb{C}$ , we have  $(F_{\mathcal{E}_N} \otimes 1)[\xi \otimes z] = [F\xi \otimes z]$ , and  $F_N[z\xi] = [zF\xi]$ .

Let us first prove that  $[\pi_N(h_N), F_N]$  is a compact operator on  $\mathcal{H}_N$ . Because  $F$  is properly supported, there is an  $f_1 \in C_c(X)$  such that  $\pi(f_1)F\pi(h) = F\pi(h)$ . Choose  $f \in C_c(X)$  such that  $f$  equals 1 on  $\text{supp } f_1 \cup \text{supp } h$ . Then  $fh = h$ , and  $\pi(f)F\pi(h) = \pi(f)\pi(f_1)F\pi(h) = \pi(f_1)F\pi(h) = F\pi(h)$ . Now

$$[\pi(p^*h_N), F] = \int_N n[\pi(h), F]n^{-1} dn, \quad (7.15)$$

by Lemma 7.3 and equivariance of  $\pi$ . Note that

$$F\pi(h) = \pi(f)F\pi(h) = \pi(f)F\pi(h)\pi(f). \quad (7.16)$$

Since  $F$ ,  $\pi(f)$  and  $\pi(h)$  are self-adjoint operators, taking adjoints in (7.16) yields  $\pi(h)F = \pi(f)\pi(h)F\pi(f)$ . Hence

$$[\pi(h), F] = \pi(f)[\pi(h), F]\pi(f),$$

and (7.15) equals  $A_f^N([\pi(h), F])$ .

By assumption, the commutator  $[\pi(h), F]$  is compact. Since  $X/N$  is compact, we can therefore apply Lemma 7.12, and conclude that  $[\pi(p^*h_N), F]$  induces a compact operator  $[\pi(p^*h_N), F]_{\mathcal{E}_N}$  on  $\mathcal{E}_N$ . Because the isomorphism  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C} \cong \mathcal{H}_N$  intertwines the compact operator  $[\pi(p^*h_N), F]_{\mathcal{E}_N} \otimes 1$  on  $\mathcal{E}_N \otimes_{C^*(N)} \mathbb{C}$  and the operator  $[\pi_N(h_N), F_N]$  on  $\mathcal{H}_N$ , the latter is compact as well.

To prove compactness of  $\pi_N(h_N)(F_N^2 - 1)$ , let  $h_N$  and  $h$  be as above. Then

$$\pi(p^*h_N)(F^2 - 1) = \int_N n\pi(h)(F^2 - 1)n^{-1} dn. \quad (7.17)$$

Because  $F$  is properly supported, so is  $F^2$ . So there is a function  $f \in C_c(X)$  such that

$$F^2\pi(h) = \pi(f)F^2\pi(h) = \pi(f)F^2\pi(h)\pi(f).$$

Taking the adjoint of this equality, we see that (7.17) equals  $A_f^N(\pi(h)(F^2 - 1))$ , which is compact. As above, this implies that  $\pi_N(h_N)(F_N^2 - 1)$  is compact.

Next, we suppose that  $N$  is compact. We saw that

$$[\pi(p^*h_N), F] = A_f^N([\pi(h), F]).$$

By Lemma 7.17 below, the operator  $A_f^N([\pi(h), F])_N$  on  $\mathcal{H}_N$  is compact. Hence the operator

$$[\pi_N(h_N), F_N] = [\pi(p^*h_N), F]_N$$

is compact as well. A similar argument can be used to prove that  $\pi_N(h_N)(F_N^2 - 1)$  is compact.

Finally, to prove that  $F_N$  is properly supported, let  $h_N \in C_c(X/N)$  and  $h \in C_c(X)$  be as above. We saw that, because  $F$  is properly supported, there is a function  $f \in C_c(X)$  such that  $\pi(f)F\pi(h) = F\pi(h)$  and  $fh = h$ . Then as before,

$$\begin{aligned} F\pi(p^*h_N) &= \int_N F n \pi(h) n^{-1} dn \\ &= \int_N n \pi(f) F \pi(h) n^{-1} dn \\ &= \int_N n \pi(f) F \pi(h) \pi(f) n^{-1} dn \\ &= A_f^N(F\pi(h)). \end{aligned}$$

Set  $K_N := p(\text{supp } f)$ , and let  $\varphi_N \in C_c(X/N)$  be equal to 1 on  $K_N$ . Then  $p^*\varphi_N f = f$ , and hence

$$\begin{aligned} \pi(p^*\varphi_N)A_f^N(F\pi(h)) &= \int_N \pi(p^*\varphi_N) n \pi(f) F \pi(h) \pi(f) n^{-1} dn \\ &= \int_N n \underbrace{\pi(n^{-1} \cdot p^*\varphi_N) \pi(f)}_{=\pi(f)} F \pi(h) \pi(f) n^{-1} dn \\ &= A_f^N(F\pi(h)). \end{aligned}$$

And therefore,

$$\pi_N(\varphi_N)F_N\pi_N(h_N) = F_N\pi_N(h_N).$$

□

In the proof of Lemma 7.16, we used the following analogue of Lemma 7.12.

**Lemma 7.17.** *Suppose  $N$  is compact. Let  $T \in \mathcal{K}(\mathcal{H})$  and  $h \in C_c(X)$  be given. Then the operator  $A_h^N(T)_N$  on  $\mathcal{H}_N$ , induced by  $A_h^N(T)$ , is compact as well.*

*Proof.* Let  $(T_j)_{j=1}^\infty$  be a sequence of finite rank operators on  $\mathcal{H}$  that converges to  $T$  in  $\mathcal{B}(\mathcal{H})$ . We first claim that the averaged operators  $A_h^N(T_j)$  have finite rank, for all  $j$ . Indeed, if  $T_j$  is a rank 1 operator:

$$T_j(\xi) = (\eta, \xi)_{\mathcal{H}} \zeta$$

for all  $\xi \in \mathcal{H}$ , then for all such  $\xi$ ,

$$\begin{aligned} A_h^N(T_j)(\xi) &= \int_N (n\pi(h)\eta, \xi)_{\mathcal{H}} n\pi(h)\zeta \, dn \\ &\subset \text{span}_{n \in N} n \cdot \pi(h)\zeta. \end{aligned}$$

By compactness of  $N$  and unitarity of the representation of  $N$  in  $\mathcal{H}$ , the unit sphere in the latter space is compact. This space is therefore finite-dimensional, so that  $A_h^N(T_j)$  is indeed a finite rank operator. In general, if  $T_j$  is a finite sum of rank 1 operators, we see that  $A_h^N(T_j)$  is still a finite rank operator.

Furthermore, we have for all  $j$ ,

$$\begin{aligned} \|A_h^N(T_j) - A_h^N(T)\|_{\mathcal{B}(\mathcal{H})} &= \left\| \int_N n\pi(h)(T_j - T)\pi(h)n^{-1} \, dn \right\|_{\mathcal{B}(\mathcal{H})} \\ &\leq \text{vol}(N) \|\pi(h)\|_{\mathcal{B}(\mathcal{H})}^2 \|T_j - T\|_{\mathcal{B}(\mathcal{H})}, \end{aligned}$$

which tends to zero. Lemma 7.14 implies that

$$\|A_h^N(T_j)_N - A_h^N(T)_N\|_{\mathcal{B}(\mathcal{H}_N)} \leq \|A_h^N(T_j) - A_h^N(T)\|_{\mathcal{B}(\mathcal{H})},$$

and we see that  $A_h^N(T_j)_N \rightarrow A_h^N(T)_N$  in  $\mathcal{B}(\mathcal{H}_N)$ .

Now the operators  $A_h^N(T_j)_N$  have finite rank. Indeed, if the image of  $A_h^N(T_j)$  is contained in the finite-dimensional subspace  $V_j \subset \mathcal{H}$ , then, since  $A_h^N(T_j)$  is properly supported,

$$A_h^N(T_j)\mathcal{H}_c \subset \mathcal{H}_c \cap V_j,$$

and the image of  $A_h^N(T_j)_N$  is contained in the (finite-dimensional) closure of  $\mathcal{H}_c \cap V_j$  in  $\mathcal{H}_N$ . It therefore follows that  $A_h^N(T)_N$  is a compact operator on  $\mathcal{H}_N$ . □

The last step in the construction of the map  $V_N$  is the fact that it is well-defined on  $K$ -homology classes:

**Lemma 7.18.** *The map  $V_N$  maps equivalent  $K$ -homology cycles to equivalent cycles.*

*Proof.* It follows from the definition of  $V_N$  that it maps unitarily equivalent cycles to unitarily equivalent cycles.

To show that  $V_N$  preserves operator homotopy, it is enough to prove that there is a constant  $C > 0$  such that for all  $K$ -homology cycles  $(\mathcal{H}, F, \pi)$  with  $F$  properly supported and  $G$ -equivariant, one has

$$\|F_N\|_{\mathcal{B}(\mathcal{H}_N)} \leq C\|F\|_{\mathcal{B}(\mathcal{H})}.$$

I.e. the map  $F \mapsto F_N$  is bounded.

For compact  $N$ , it follows from Lemma 7.14 that  $\|F_N\|_{\mathcal{B}(\mathcal{H}_N)} \leq \|F\|_{\mathcal{B}(\mathcal{H})}$ , and we are done. Therefore, suppose that  $X/N$  is compact.

Let  $(\mathcal{H}, F, \pi)$  be a  $K$ -homology cycle over  $X$ , with  $F$  properly supported and  $G$ -equivariant. As before, let  $\mathcal{E}_N$  be the completion of  $\mathcal{H}_c$  in the inner product (7.12). By Lemma 7.7,  $F$  induces a bounded operator  $F_{\mathcal{E}_N}$  on  $\mathcal{E}_N$ , and by (7.8) we have

$$\begin{aligned} \|F_{\mathcal{E}_N}\|_{\mathcal{B}(\mathcal{E}_N)} &= \|SF_{\mathcal{E}_N}S^*\|_{\mathcal{B}(L^2(N, \mathcal{H}))} \\ &= \left\| \int_N \lambda^N(n) \otimes \pi(f)F\pi(n \cdot f)n \, dn \right\|, \end{aligned}$$

where  $f \in C_c(X)$  has the property that  $\int_N f(nx)^2 \, dn = 1$  for all  $x \in X$ . Because  $\lambda^N(n)$  and  $n$  are unitary operators on  $L^2(N)$  and  $\mathcal{H}$  respectively, this norm is at most equal to

$$\int_K \|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \|\pi(n \cdot f)\|_{\mathcal{B}(\mathcal{H})} \|F\|_{\mathcal{B}(\mathcal{H})} \, dn = \text{vol}(K) \|\pi(f)\|_{\mathcal{B}(\mathcal{H})}^2 \|F\|_{\mathcal{B}(\mathcal{H})},$$

where  $K$  is the compact set  $\{n \in N; f \cdot nf \neq 0\}$ . Set

$$C := \text{vol}(K) \|\pi(f)\|_{\mathcal{B}(\mathcal{H})}^2,$$

so that  $\|F_{\mathcal{E}_N}\|_{\mathcal{B}(\mathcal{E}_N)} \leq C\|F\|_{\mathcal{B}(\mathcal{H})}$ .

Then for all  $\xi \in \mathcal{H}_c$ , we have  $\|F_{\mathcal{E}_N}\xi\|_{\mathcal{E}_N} \leq C\|F\|_{\mathcal{B}(\mathcal{H})} \|\xi\|_{\mathcal{E}_N}$ . Therefore, as in (7.13), we see that

$$C^2 \|F\|_{\mathcal{B}(\mathcal{H})}^2 (\xi, \xi)_{C^*(N)} - (F_{\mathcal{E}_N}\xi, F_{\mathcal{E}_N}\xi)_{C^*(N)}$$

is a positive element of  $C^*(N)$ . Applying the trivial representation, we conclude that

$$C^2 \|F\|_{\mathcal{B}(\mathcal{H})}^2 \|\xi\|_N^2 - \|F\xi\|_N^2 \geq 0$$

for all  $\xi \in \mathcal{H}_c$ , i.e.  $\|F_N\|_{\mathcal{B}(\mathcal{H}_N)} \leq C\|F\|_{\mathcal{B}(\mathcal{H})}$ . □

## 7.5 Proof of naturality of the assembly map

Having finished the construction of the homomorphism  $V_N$ , we are now ready to prove Theorem 7.1.

*Proof of Theorem 7.1.*

*Step 1: the  $K_0^{G/N}(X/N)$ -cycles.* Let  $[\mathcal{H}, F, \pi] \in K_0^G(X)$ , and suppose  $F$  is  $G$ -equivariant and properly supported. Our goal is to show that

$$(\int_N)_* \circ \mu_X^G[\mathcal{H}, F, \pi] = \mu_{X/N}^{G/N}[\mathcal{H}_N, F_N, \pi_N]$$

as elements of  $K_0(C^*(G/N))$ .

Let  $\mathcal{E}$  and  $F_{\mathcal{E}}$  be the Hilbert  $C^*(G)$ -module and the operator on  $\mathcal{E}$  constructed from the cycle  $(\mathcal{H}, F, \pi)$  as in the definition of the assembly map. That is,

$$\mu_X^G[\mathcal{H}, F, \pi] = [\mathcal{E}, F_{\mathcal{E}}].$$

The Hilbert  $C^*(G/N)$ -module part of  $(\int_N)_* \circ \mu_X^G[\mathcal{H}, F, \pi]$  is

$$\mathcal{E}_{G/N} := \mathcal{E} \otimes_{C^*(G)} C^*(G/N),$$

where  $C^*(G)$  acts on  $C^*(G/N)$  via the homomorphism  $\int_N$ . The  $C^*(G/N)$ -valued inner product on  $\mathcal{E}_{G/N}$  is given by

$$([\xi \otimes a], [\eta \otimes b])_{\mathcal{E}_{G/N}} = a^* (\int_N ((\xi, \eta)_{\mathcal{E}})) b,$$

for all  $\xi, \eta \in \mathcal{E}$  and  $a, b \in C^*(G/N)$ . The operator part of  $(\int_N)_* \circ \mu_X^G[\mathcal{H}, F, \pi]$  is

$$F_{\mathcal{E}_{G/N}} := F_{\mathcal{E}} \otimes 1.$$

On the other hand, the Hilbert  $C^*(G/N)$ -module part of  $\mu_{X/N}^{G/N}[\mathcal{H}_N, F_N, \pi_N]$  is a certain completion  $\tilde{\mathcal{E}}_{G/N}$  of the space

$$\mathcal{H}_{N,c} = \pi_N(C_c(X/N)) \mathcal{H}_N.$$

The completion  $\tilde{\mathcal{E}}_{G/N}$  of  $\mathcal{H}_{N,c}$  is taken in the norm

$$\|\xi_N\|_{\tilde{\mathcal{E}}_{G/N}}^2 = \|Ng \mapsto (\xi_N, Ng \cdot \xi_N)_N\|_{C^*(G/N)}.$$

The operator part  $F_{\tilde{\mathcal{E}}_{G/N}}$  of  $\mu_{X/N}^{G/N}[\mathcal{H}_N, F_N, \pi_N]$  is defined as the continuous extension of  $F_N$ , as in Lemma 7.7.

*Step 2: an isomorphism.* If  $\xi \in \mathcal{H}_c$ , we will write  $\xi^N := \xi + \ker(-, -)_N$  for its class in  $\mathcal{H}_N$ . Then for all  $\xi \in \mathcal{H}_c$ , we have  $\xi^N \in \mathcal{H}_{N,c}$ . Indeed, let  $f \in C_c(X)$  and  $\zeta \in \mathcal{H}$  be such that  $\xi = \pi(f)\zeta$ . Let  $h_N \in C_c(X/N)$  be equal to 1 on the image of  $\text{supp } f$  in  $X/N$ . Then

$$\begin{aligned} \xi^N &= \pi(p^* h_N) \pi(f) \zeta + \ker(-, -)_N \\ &= \pi_N(h_N) (\pi(f) \zeta + \ker(-, -)_N) \\ &= \pi_N(h_N) \xi^N. \end{aligned}$$

Define the linear map

$$\Psi : \mathcal{H}_c \otimes_{C_c(G)} C_c(G/N) \rightarrow \mathcal{H}_{N,c}$$

by

$$\Psi[\xi \otimes \varphi] = \int_{G/N} \varphi(Ng^{-1}) Ng \cdot \xi^N d(Ng),$$

where  $d(Ng)$  is the Haar measure on  $G/N$  normalised such that<sup>1</sup> for all  $h \in C_c(G)$ ,

$$\int_G h(g) dg = \int_{Ng \in G/N} \int_{n \in N} h(gn) dn d(Ng). \quad (7.18)$$

---

<sup>1</sup>The correct way to define the integral on the right hand side of (7.18) is via a measurable section  $G/N \rightarrow G$ .

We will show that  $\Psi$  is an isometry with respect to the  $C_c(G/N) \subset C^*(G/N)$ -valued inner products on the spaces in question. This implies that  $\Psi$  extends to an isometry between the completions in these inner products:

$$\Psi : \mathcal{E}_{G/N} = \overline{\mathcal{H}_c \otimes_{C_c(G)} C_c(G/N)} \rightarrow \overline{\mathcal{H}_{N,c}} = \tilde{\mathcal{E}}_{G/N}.$$

It will turn out that  $\Psi$  is surjective, and intertwines the operators  $F_{\mathcal{E}_{G/N}}$  and  $F_{\tilde{\mathcal{E}}_{G/N}}$ . This will complete the proof.

To prove that  $\Psi$  is an isometry, let  $\xi, \eta \in \mathcal{H}_c$  and  $\varphi, \psi \in C_c(G/N)$  be given. Then for all  $g \in G$ , one computes

$$\begin{aligned} ([\xi \otimes \varphi], [\eta \otimes \psi])_{\mathcal{E}_{G/N}}(Ng) &= \\ \int_{G/N} \int_{G/N} \bar{\varphi}(Ng'^{-1}) \psi(Ng''^{-1}g'^{-1}g) (\xi, g'' \cdot \xi)_N d(Ng') d(Ng'') &= \\ (\Psi[\xi \otimes \varphi], \Psi[\eta \otimes \psi])_{\tilde{\mathcal{E}}_{G/N}}(Ng). \end{aligned} \quad (7.19)$$

Next, we show that  $\Psi : \mathcal{E}_{G/N} \rightarrow \tilde{\mathcal{E}}_{G/N}$  has dense image, and is hence surjective, because it is an isometry. Indeed, let  $\xi \in \mathcal{H}_c$ . We will show that  $\xi^N$  lies in the closure of the image of  $\Psi$ . Because  $\mathcal{H}_c / \ker(-, -)_N$  is dense in  $\mathcal{H}_{N,c}$ , which in turn is dense in  $\tilde{\mathcal{E}}_{G/N}$ , this proves that  $\Psi$  has dense image. Let us construct a sequence in  $\mathcal{H}_c \otimes_{C_c(G)} C_c(G/N)$  whose image under  $\Psi$  converges to  $\xi^N$ . Let  $(\varphi_N^j)_{j=1}^\infty$  be a sequence in  $C_c(G/N)$  such that for all  $j$ ,  $\varphi_N^j$  is a nonnegative real valued function with integral 1, and that

$$\lim_{j \rightarrow \infty} \varphi_N^j = \delta_{Ne},$$

as distributions on  $G/N$  (with respect to the Haar measure  $d(Ng)$ ). Then for all  $j$ ,

$$\begin{aligned} \|\Psi(\xi \otimes \varphi_N^j) - \xi^N\|_N &= \\ \left\| \int_{N/G} Ng \cdot \xi^N \varphi_N^j(Ng^{-1}) d(Ng) - \int_{N/G} \xi^N \varphi_N^j(Ng^{-1}) d(Ng) \right\|_N &\leq \\ \int_{N/G} \varphi_N^j(Ng^{-1}) \|Ng \cdot \xi^N - \xi^N\| d(Ng), \end{aligned}$$

which tends to zero as  $j \rightarrow \infty$ . So  $\Psi$  is surjective.

Finally, it follows directly from the definitions that  $\Psi \circ (F_{\mathcal{E}} \otimes 1) = F_{\tilde{\mathcal{E}}_{G/N}} \circ \Psi$ .  $\square$

# Chapter 8

## $K$ -homology classes of differential operators

In this chapter, we will compute the image under the homomorphism  $V_N$  from Theorem 7.1 of a  $K$ -homology class associated to an equivariant elliptic differential operator on a vector bundle over a smooth manifold. The result is Corollary 8.11. In Chapter 10, we will specialise Corollary 8.11 to Dirac operators in the case of a free action by a discrete group, proving Theorem 6.5. Corollary 8.11 will also play a role in Section 9.4.

### 8.1 $L^2$ -spaces of sections of a vector bundle

Let  $G$  be a unimodular Lie group with a Haar measure  $dg$ , and let  $N$  be a closed, normal subgroup of  $G$ , with a left invariant Haar measure  $dn$ . Let  $M$  be a smooth manifold on which  $G$  acts properly, such that the action of  $N$  on  $M$  is free. Suppose  $M/G$  is compact.

Now let  $q : E \rightarrow M$  be a  $G$ -vector bundle, equipped with a  $G$ -invariant Hermitian metric  $(-, -)_E$ . Let  $dm$  be a  $G$ -invariant measure on  $M$ , and let  $L^2(M, E)$  be the space of square-integrable sections of  $E$  with respect to this measure. Let  $\pi^M : C_0(M) \rightarrow \mathcal{B}(L^2(M, E))$  be the representation defined by multiplying sections with functions. Let  $L^2(M, E)_N$  be the Hilbert space constructed from  $L^2(M, E)$  as in the definition of the homomorphism  $V_N$ . We will show that  $L^2(M, E)_N$  is  $G/N$ -equivariantly and unitarily isomorphic to the Hilbert space  $L^2(M/N, E/N)$  of square-integrable sections of the quotient vector bundle

$$q_N : E/N \rightarrow M/N.$$

The  $L^2$ -inner product on sections of  $E/N$  is defined via the metric on  $E/N$  induced by the one on  $E$ , and the measure  $d\mathcal{O}$  on  $M/N$  with the property that for all measurable sections<sup>1</sup>  $\varphi : M/N \rightarrow M$  and all  $f \in C_c(M)$ ,

$$\int_M f(m) dm = \int_{M/N} \int_N f(n \cdot \varphi(\mathcal{O})) dn d\mathcal{O} \quad (8.1)$$

(see [13], Proposition 4b, page 44).

---

<sup>1</sup>Measurable in the sense that the inverse image of any Borel measurable subset of  $M$  is Borel measurable in  $M/N$ .



Note that in this example, the space

$$L_c^2(M, E) := \pi(C_c(M))L^2(M, E)$$

is the space of compactly supported  $L^2$ -sections of  $E$ . Consider the linear map

$$\chi : L_c^2(M, E) \rightarrow L^2(M/N, E/N), \quad (8.2)$$

defined by

$$\chi(s)(Nm) := N \cdot \int_N n \cdot s(n^{-1}m) dn,$$

for all  $s \in L_c^2(M, E)$  and  $m \in M$ . Because  $s$  is compactly supported and the action is proper, the integrand is compactly supported for all  $m \in M$ .

**Proposition 8.1.** *The map  $\chi$  induces a  $G/N$ -equivariant unitary isomorphism*

$$\chi : L^2(M, E)_N \xrightarrow{\cong} L^2(M/N, E/N). \quad (8.3)$$

*Proof.* It follows from a lengthy but straightforward computation that the map  $\chi$  is isometric, in the sense that for all  $s \in L_c^2(M, E)$ ,

$$\|\chi(s)\|_{L^2(M/N, E/N)} = \|s\|_N,$$

where  $\|\cdot\|_N$  is the norm corresponding to the inner product  $(-, -)_N$ . Hence  $\chi$  induces an injective linear map

$$\chi : L_c^2(M, E)/\mathcal{K} \rightarrow L^2(M/N, E/N), \quad (8.4)$$

where  $\mathcal{K}$  is the space of sections  $s \in L_c^2(M, E)$  with  $\|s\|_N = 0$ .

Furthermore, the map  $\chi$  has dense image, see Lemma 8.2 below. It therefore extends to a unitary isomorphism

$$\chi : L^2(M, E)_N \rightarrow L^2(M/N, E/N).$$

The fact that  $N$  is a normal subgroup implies that this isomorphism intertwines the pertinent representations of  $G/N$ .  $\square$

**Lemma 8.2.** *The image of the map  $\chi$  in (8.2) contains the space  $L_c^2(M/N, E/N)$  of compactly supported  $L^2$ -sections of  $E/N \rightarrow M/N$ .*

*Proof.* Let  $\sigma \in L_c^2(M/N, E/N)$ . We will construct a section  $s \in L_c^2(M, E)$  such that  $\chi(s) = \sigma$ , using the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{p_E} & E/N \\ \downarrow q & & \downarrow q_N \\ M & \xrightarrow{p} & M/N. \end{array}$$

Here the horizontal maps are quotient maps and define principal fibre bundles, and the vertical maps are vector bundle projections.

Let  $\{U_j\}$  be an open cover of  $\text{supp } \sigma \subset M/N$  that admits local trivialisations

$$\begin{aligned}\tau_j &: p^{-1}(U_j) \xrightarrow{\cong} U_j \times N; \\ \theta_j^N &: q_N^{-1}(U_j) \xrightarrow{\cong} U_j \times E_0,\end{aligned}$$

where  $E_0$  is the typical fibre of  $E$ . Because  $\text{supp } \sigma$  is compact, the cover  $\{U_j\}$  may be supposed to be finite. Via the isomorphism of vector bundles  $p^*(E/N) \cong E$ , the trivialisations  $\theta_j^N$  induce local trivialisations of  $E$ :

$$\theta_j : q^{-1}(p^{-1}(U_j)) \xrightarrow{\cong} p^{-1}(U_j) \times E_0.$$

And then, we can form trivialisations

$$\tau_j^E : p_E^{-1}(q_N^{-1}(U_j)) \xrightarrow{\cong} q_N^{-1}(U_j) \times N,$$

by

$$\begin{aligned}p_E^{-1}(q_N^{-1}(U_j)) &= q^{-1}(p^{-1}(U_j)) \\ &\cong p^{-1}(U_j) \times E_0 && \text{via } \theta_j \\ &\cong U_j \times N \times E_0 && \text{via } \tau_j \\ &\cong q_N^{-1}(U_j) \times N && \text{via } \theta_j^N.\end{aligned}$$

Here the symbol ' $\cong$ ' indicates an  $N$ -equivariant diffeomorphism. It follows from the definition of the trivialisation  $\theta_j$  that  $\tau_j^E$  composed with projection onto  $q_N^{-1}(U_j)$  equals  $p_E$ , so that  $\tau_j^E$  is indeed an isomorphism of principal  $N$ -bundles.

For every  $j$ , define the section  $s_j \in L^2(M, E)$  by

$$s_j(\tau_j^{-1}(\mathcal{O}, n)) = (\tau_j^E)^{-1}(\sigma(\mathcal{O}), n)$$

for all  $\mathcal{O} \in U_j$  and  $n \in N$ , and extended by zero outside  $p^{-1}(U_j)$ . By compactness of  $\text{supp } \sigma$ , there is a compact subset  $\tilde{C} \subset M$  that intersects all  $N$ -orbits in  $\text{supp } \sigma$ . Let  $K \subset N$  be a compact subset of  $dn$ -volume 1, and set  $C := K \cdot \tilde{C}$ . Then for all  $m \in M$ , the volume of the compact set

$$V_m := \{n \in N; n^{-1}m \in C\}$$

is at least 1. Define the section  $\tilde{s}$  of  $E$  by

$$\tilde{s}(m) = \begin{cases} \sum_j s_j(m) & \text{if } m \in C \\ 0 & \text{if } m \notin C. \end{cases}$$

Then  $\tilde{s} \in L_c^2(M, E)$ , and for all  $m \in M$ ,

$$\begin{aligned}\chi(\tilde{s})(Nm) &= \sum_{\substack{j, \\ Nm \in U_j}} \int_{V_m} p_E(n \cdot s_j(n^{-1}m)) \, dn \\ &= \sum_{\substack{j, \\ Nm \in U_j}} \int_{V_m} p_E((\tau_j^E)^{-1}(\sigma(Nm), n \cdot \psi(n^{-1}m))) \, dn,\end{aligned} \tag{8.5}$$

where  $(Nm, \psi(n^{-1}m)) := \tau_j(n^{-1}m)$ . Now since  $p_E \circ (\tau_j^E)^{-1}$  is projection onto  $q_N^{-1}(U_j)$ , the expression (8.5) equals

$$\#\{j; Nm \in U_j\} \text{vol}(V_m) \sigma(Nm).$$

Setting  $\varphi(m) := \#\{j; Nm \in U_j\} \text{vol}(V_m)$  gives a measurable function  $\varphi$  on  $M$  which is bounded below by 1 and  $N$ -invariant by invariance of  $dn$ . Hence

$$s := \frac{1}{\varphi} \tilde{s},$$

is a section  $s \in L_c^2(M, E)$  for which  $\chi(s) = \sigma$ . □

## 8.2 Differential operators

Let  $G$  and  $E \rightarrow M$  be as in Section 8.1. Let  $D : \Gamma^\infty(M, E) \rightarrow \Gamma^\infty(M, E)$  be a  $G$ -equivariant first order differential operator that is symmetric with respect to the  $L^2$ -inner product on compactly supported sections. Then  $D$  defines an unbounded operator on  $L^2(M, E)$ . We assume that this operator has a self-adjoint extension, which we also denote by  $D$ .

### Functional calculus and properly supported operators

Applying the functional calculus to the self-adjoint extension of  $D$ , we define the bounded, self-adjoint operator  $b(D)$  on  $L^2(M, E)$ , for any bounded measurable function  $b$  on  $\mathbb{R}$ . The operator  $b(D)$  is  $G$ -equivariant because of Lemma 4.31.

We will later consider the case where  $(L^2(M, E), b(D), \pi^M)$  is a  $K$ -homology cycle, and apply the map  $V_N$  to this cycle. It is therefore important to us that the operator  $b(D)$  is properly supported (Definition 5.13) for well-chosen functions  $b$ :

**Proposition 8.3.** *If  $b$  is a bounded measurable function with compactly supported (distributional) Fourier transform  $\hat{b}$ , then the operator  $b(D)$  is properly supported.*

The proof of this proposition is based on the following two facts, whose proofs can be found in [34], Section 10.3.

**Proposition 8.4.** *If  $b$  is a bounded measurable function on  $\mathbb{R}$  with compactly supported Fourier transform, then for all  $s, t \in \Gamma_c^\infty(M, E)$ ,*

$$(b(D)s, t)_{L^2(M, E)} = \frac{1}{2\pi} \int_{\mathbb{R}} (e^{i\lambda D}s, t)_{L^2(M, E)} \hat{b}(\lambda) d\lambda.$$

This is Proposition 10.3.5. from [34]. By Stone's theorem, the operator  $e^{i\lambda D}$  is characterised by the requirements that  $\lambda \mapsto e^{i\lambda D}$  is a group homomorphism from  $\mathbb{R}$  to the unitary operators on  $L^2(M, E)$ , and that for all  $s \in \Gamma_c^\infty(M, E)$ ,

$$\left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} e^{i\lambda D}s = iDs.$$

**Lemma 8.5.** *Let  $s \in \Gamma_c^\infty(M, E)$ , and let  $h \in C_c^\infty(M)$  be equal to 1 on the support of  $s$ . Let  $\lambda \in \mathbb{R}$  such that  $|\lambda| < \|[D, \pi^M(h)]\|^{-1}$ . Then*

$$\text{supp } e^{i\lambda D}s \subset \text{supp } h.$$

This follows from the proof of Proposition 10.3.1. from [34].

*Proof of Proposition 8.3.* Let  $R > 0$  be such that  $\text{supp } \hat{b} \subset [-R, R]$ . Let  $f \in C_c(M)$ , and choose  $h \in C_c^\infty(M)$  such that  $h$  equals 1 on the support of  $f$ , and that  $\|[D, \pi^M(h)]\| \leq \frac{1}{R}$ . Let  $1_M$  be the constant function 1 on  $M$ . Then by Lemma 8.5,

$$\pi^M(1_M - h)e^{i\lambda D}\pi^M(f) = 0, \quad (8.6)$$

for all  $\lambda \in ]-R, R[$ . Here we have extended the nondegenerate representation  $\pi^M$  of  $C_0(M)$  on  $L^2(M, E)$  to the multiplier algebra  $C_b(M)$  of  $C_0(M)$ . So by Proposition 8.4, we have for all  $s, t \in \Gamma_c^\infty(M, E)$ ,

$$\begin{aligned} (\pi^M(1_M - h)b(D)\pi^M(f)s, t)_{L^2(M, E)} &= (b(D)\pi^M(f)s, \pi^M(1_M - \bar{h})t)_{L^2(M, E)} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} (e^{i\lambda D}\pi^M(f)s, \pi^M(1_M - \bar{h})t)_{L^2(M, E)} \hat{b}(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{-R}^R (\pi^M(1_M - h)e^{i\lambda D}\pi^M(f)s, t)_{L^2(M, E)} \hat{b}(\lambda) d\lambda \\ &= 0, \end{aligned}$$

by (8.6). So

$$(1 - \pi^M(h))b(D)\pi^M(f) = \pi^M(1_M - h)b(D)\pi^M(f) = 0,$$

and hence  $b(D)$  is properly supported.  $\square$

### The image of $b(D)$ under $V_N$

Now suppose that  $D$  is elliptic and that  $b$  is a normalising function with compactly supported Fourier transform. If  $g$  is a smooth, even, compactly supported function on  $\mathbb{R}$ , and  $f := g * g$  is its convolution square, then  $b(\lambda) := \int_{\mathbb{R}} \frac{e^{i\lambda x} - 1}{ix} f(x) dx$  is such a function (see [34], Exercise 10.9.3).

Because  $b(D)$  is properly supported it preserves  $L_c^2(M, E)$ , and the construction used in the definition of the map  $V_N$  applies to  $b(D)$ . The resulting operator  $b(D)_N$  on  $L^2(M, E)_N$  is defined by commutativity of the following diagram:

$$\begin{array}{ccc} L_c^2(M, E) & \longrightarrow & L^2(M, E)_N \\ \downarrow b(D) & & \downarrow b(D)_N \\ L_c^2(M, E) & \longrightarrow & L^2(M, E)_N. \end{array}$$

On the other hand, the operator  $D$  induces an unbounded operator on  $L^2(M/N, E/N)$ , because it restricts to

$$\tilde{D}^N : \Gamma^\infty(M, E)^N \rightarrow \Gamma^\infty(M, E)^N.$$

We then use the following fact:

**Proposition 8.6.** *Let  $H$  be a group acting properly and freely on a manifold  $M$ . Let  $q : E \rightarrow M$  be an  $H$ -vector bundle. Then the induced projection*

$$q^H : E/H \rightarrow M/H$$

*defines a vector bundle over  $M/H$ .*

*Let  $\Gamma^\infty(M, E)^H$  be the space of  $H$ -invariant sections of  $E$ . The linear map*

$$\psi_E : \Gamma^\infty(M, E)^H \rightarrow \Gamma^\infty(M/H, E/H), \quad (8.7)$$

*defined by*

$$\psi_E(s)(H \cdot m) = H \cdot s(m),$$

*is an isomorphism of  $C^\infty(M)^H \cong C^\infty(M/H)$ -modules.*

*Sketch of proof.* The inverse of  $\psi_E$  is the pullback along the quotient map  $p : M \rightarrow M/H$ ,

$$p^* : \Gamma^\infty(M/H, E/H) \rightarrow \Gamma^\infty(M, E)^H,$$

*defined by*

$$(p^*\sigma)(m) = (m, \sigma(Hm)) \in p^*(E/H) \cong E,$$

*for  $\sigma \in \Gamma^\infty(M/H, E/H)$ . The isomorphism  $p^*(E/H) \cong E$  is given by*

$$(m, He) \mapsto e, \quad (8.8)$$

*for  $m \in M$  and  $e \in E_m$ .* □

Using Proposition 8.6, we define

$$D^N := \psi_E^{-1} \tilde{D}^N \psi_E : \Gamma^\infty(M/N, E/N) \rightarrow \Gamma^\infty(M/N, E/N). \quad (8.9)$$

We regard  $D^N$  as an unbounded operator on  $L^2(M/N, E/N)$ . It is symmetric with respect to the  $L^2$ -inner product, and hence essentially self-adjoint by [34], Corollary 10.2.6. We therefore have the bounded operator  $b(D^N)$  on  $L^2(M/N, E/N)$ .

Our claim is:

**Proposition 8.7.** *The isomorphism  $\chi$  from Proposition 8.1 intertwines the operators  $b(D)_N$  and  $b(D^N)$ :*

$$\begin{array}{ccc} L^2(M, E)_N & \xrightarrow{\chi} & L^2(M/N, E/N) \\ \downarrow b(D)_N & & \downarrow b(D^N) \\ L^2(M, E)_N & \xrightarrow{\chi} & L^2(M/N, E/N). \end{array}$$

We will prove this claim by reducing it to the commutativity of another diagram. This diagram involves the Hilbert space  $\tilde{L}^2(M/N, E/N)$ , which is defined as the completion of the space  $\Gamma^\infty(M, E)^N$  in the inner product

$$(\sigma, \tau) := \int_{M/N} (\sigma(\varphi(\mathcal{O})), \tau(\varphi(\mathcal{O})))_E d\mathcal{O},$$

for any measurable section  $\varphi : M/N \rightarrow M$ . The map  $\psi_E$  from Proposition 8.6 extends continuously to a unitary isomorphism

$$\tilde{\psi}_E : \tilde{L}^2(M/N, E/N) \rightarrow L^2(M/N, E/N).$$

The unbounded operator  $\tilde{D}^N$  on  $\tilde{L}^2(M/N, E/N)$  is essentially self-adjoint because  $D^N$  is, and because  $\tilde{\psi}_E$  intertwines the two operators. Hence we have  $b(\tilde{D}^N) \in \mathcal{B}(\tilde{L}^2(M/N, E/N))$ . We will deduce Proposition 8.7 from Lemma 8.8:

**Lemma 8.8.** *The following diagram commutes:*

$$\begin{array}{ccc} L_c^2(M, E) & \xrightarrow{\int_N n \cdot} & \tilde{L}^2(M/N, E/N) \\ \downarrow b(D) & & \downarrow b(\tilde{D}^N) \\ L_c^2(M, E) & \xrightarrow{\int_N n \cdot} & \tilde{L}^2(M/N, E/N), \end{array}$$

where the map  $\int_N n \cdot$  is given by<sup>2</sup>

$$\left( \int_N n \cdot (s) \right) (Nm) = \int_N n \cdot s(n^{-1}m) dn.$$

*Proof. Step 1.* Because the representation of  $N$  in  $L^2(M, E)$  is unitary, we have

$$\left( \int_N n \cdot (s), t \right)_{L^2(M, E)} = \left( s, \int_N n \cdot (t) \right)_{L^2(M, E)}$$

for all  $s, t \in L_c^2(M, E)$ .

*Step 2.* By equivariance of  $D$ , we have

$$\left( \int_N n \cdot \right) \circ D = \tilde{D}^N \circ \int_N n \cdot$$

on  $\Gamma_c^\infty(M, E)$ .

*Step 3.* For all  $s \in \Gamma_c^\infty(M, E)$ , we have

$$\begin{aligned} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} \int_N n \cdot \circ e^{i\lambda D} s &= \int_N \frac{\partial}{\partial \lambda} e^{i\lambda D} n \cdot s dn \\ &= i \int_N n \cdot D s dn \\ &= i \tilde{D}^N \int_N n \cdot (s) \quad (\text{by Step 2}) \\ &= \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} e^{i\lambda \tilde{D}^N} \int_N n \cdot (s). \end{aligned}$$

So by Stone's theorem,

$$\int_N n \cdot \circ e^{i\lambda D} = e^{i\lambda \tilde{D}^N} \circ \int_N n \cdot$$

---

<sup>2</sup>Note that the space  $\tilde{L}^2(M/N, E/N)$  can be realised as a space of sections of  $E$ .

for all  $\lambda \in \mathbb{R}$ .

*Step 4.* By using Proposition 8.4 and Steps 1 and 3 several times, we finally see that for all  $s, t \in \Gamma_c^\infty(M, E)$ ,

$$\begin{aligned}
 (b(\tilde{D}^N) \int_N n \cdot (s), t)_{L^2(M, E)} &= \frac{1}{2\pi} \int_{\mathbb{R}} (e^{i\lambda \tilde{D}^N} \int_N n \cdot (s), t)_{L^2(M, E)} \hat{b}(\lambda) d\lambda \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} (\int_N n \cdot e^{i\lambda D} s, t)_{L^2(M, E)} \hat{b}(\lambda) d\lambda \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} (e^{i\lambda D} s, \int_N n \cdot (t))_{L^2(M, E)} \hat{b}(\lambda) d\lambda \\
 &= (b(\tilde{D}) s, \int_N n \cdot (t))_{L^2(M, E)} \\
 &= (\int_N n \cdot b(\tilde{D}) s, t)_{L^2(M, E)}.
 \end{aligned}$$

This completes the proof.  $\square$

We now derive Proposition 8.7 from Lemma 8.8.

*Proof of Proposition 8.7.* Consider the following cube:

$$\begin{array}{ccccc}
 L_c^2(M, E) & \xrightarrow{\int_N n \cdot} & \tilde{L}^2(M/N, E/N) & & \\
 b(D) \downarrow & & b(\tilde{D}^N) \downarrow & \searrow \tilde{\psi}_E & \\
 L_c^2(M, E) & \longrightarrow & \tilde{L}^2(M/N, E/N) & & \\
 & \searrow & \downarrow \tilde{\psi}_E & \searrow \tilde{\psi}_E & \\
 & & L^2(M, E)_N & \xrightarrow{\chi} & L^2(M/N, E/N) \\
 & & \downarrow b(D)_N & & \downarrow b(D^N) \\
 & & L^2(M, E)_N & \xrightarrow{\chi} & L^2(M/N, E/N).
 \end{array}$$

The rear square (with the operators  $b(D)$  and  $b(\tilde{D}^N)$  in it) commutes by Lemma 8.8. The left hand square (with the operators  $b(D)$  and  $b(D)_N$ ) commutes by definition of  $b(D)_N$ , and the right hand square (with  $b(\tilde{D}^N)$  and  $b(D^N)$ ) commutes by Lemma 4.31. The top and bottom squares commute by definition of the map  $\chi$ , so that the front square commutes as well, which is Proposition 8.7.  $\square$

### 8.3 Multiplication of sections by functions

Let  $G, M$  and  $E$  be as in Sections 8.1 and 8.2. As before, let

$$\pi^M : C_0(M) \rightarrow \mathcal{B}(L^2(M, E))$$

and

$$\pi^{M/N} : C_0(M/N) \rightarrow \mathcal{B}(L^2(M/N, E/N))$$

be the representations defined by multiplication of sections by functions. Let

$$\pi_N^M : C_0(M/N) \rightarrow \mathcal{B}(L^2(M, E)_N)$$

be the representation obtained from  $\pi^M$  by the procedure in Section 7.4.

**Lemma 8.9.** *The isomorphism (8.3) intertwines the representations  $\pi_N^M$  and  $\pi^{M/N}$ .*

*Proof.* The representation  $\pi_N^M$  is induced by

$$\begin{aligned} (\pi^M)^N : C(M/N) &\rightarrow \mathcal{B}(L_c^2(M, E)), \\ (\pi^M)^N(f)s(m) &= f(N \cdot m)s(m). \end{aligned}$$

For all  $f \in C(M/N)$ ,  $s \in L_c^2(M, E)$  and  $m \in M$ , we therefore have

$$\begin{aligned} \chi(\pi_N^M(f)s)(N \cdot m) &= \chi\left((\pi^M)^N(f)s\right)(N \cdot m) \\ &= N \cdot \int_N n \cdot f(N \cdot n^{-1}m)s(n^{-1} \cdot m) dn \\ &= N \cdot f(N \cdot m) \int_N n \cdot s(n^{-1} \cdot m) dn \\ &= \left(\pi^{M/N}(f)\chi(s)\right)(N \cdot m). \end{aligned}$$

□

## 8.4 Conclusion

Let  $G, M, E, D, D^N, \pi^M$  and  $\pi^{M/N}$  be as in Sections 8.1 – 8.3. Suppose that the vector bundle  $E$  carries a  $\mathbb{Z}_2$ -grading with respect to which the operator  $D$  is odd. Suppose  $D$  is elliptic and essentially self-adjoint as an unbounded operator on  $L^2(M, E)$ .<sup>3</sup> Let  $b$  be a normalising function with compactly supported Fourier transform. Then Proposition 8.1, Proposition 8.7 and Lemma 8.9 may be summarised as follows.

**Theorem 8.10.** *Let  $(L^2(M, E)_N, b(D)_N, \pi_N^M)$  be the triple obtained from  $(L^2(M, E), b(D), \pi^M)$  by the procedure of Section 7.4. Then there is a unitary isomorphism*

$$\chi : L^2(M, E)_N \rightarrow L^2(M/N, E/N)$$

*that intertwines the representations of  $G/N$ , the operators  $b(D)_N$  and  $b(D^N)$ , and the representations  $\pi_N^M$  and  $\pi^{M/N}$ .*

**Corollary 8.11.** *The image of the class*

$$[D] := [L^2(M, E), b(D), \pi^M] \in K_0^G(M)$$

*under the homomorphism  $V_N$  defined in Section 7.4 is*

$$V_N[D] = [L^2(M/N, E/N), b(D^N), \pi^{M/N}] = [D^N] \in K_0^{G/N}(M/N).$$

---

<sup>3</sup>This is the case if  $M$  is complete and  $D$  is a Dirac operator on  $M$ , see Corollary 4.36.



**Remark 8.12.** If the action of  $G$  on  $M$  happens to be free, then Corollary 8.11 allows us to restate the Guillemin–Sternberg–Landsman conjecture 6.4 without using techniques from non-commutative geometry. Indeed, for free actions we have

$$\begin{aligned} R_G^0 \circ \mu_M^G[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*] &= \mu_{M/G}^{\{e\}} \circ V_G[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*] && \text{(by Theorem 7.1)} \\ &= \text{index}(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^G && \text{(by Corollary 8.11)} \\ &= \dim\left(\ker(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^+\right)^G - \dim\left(\ker(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^-\right)^G. \end{aligned}$$

Note that even though the vector spaces  $\ker(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\pm$  may be infinite-dimensional, their  $G$ -invariant parts are not, because they are the kernels of the elliptic operators  $\left((\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\pm\right)^G$  on the compact manifold  $M/G$ . So Conjecture 6.4 becomes

$$\dim\left(\ker(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^+\right)^G - \dim\left(\ker(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^-\right)^G = \text{index } \bar{\partial}_{L^{\omega_0}} + \bar{\partial}_{L^{\omega_0}}^*.$$

In the setting of Theorem 6.5, the assumption that the action is free is a very restrictive one, see Remark 6.6.

# Chapter 9

## Inclusions of maximal compact subgroups into semisimple groups

The monomorphism part of Valette’s ‘naturality of the assembly map’ is harder to generalise to nondiscrete groups than the epimorphism part (Theorem 7.1). The reason for this is more or less that the geometry of homogeneous spaces of nondiscrete groups is usually nontrivial. More specifically, the problem is that a principal fibre bundle  $G \rightarrow G/H$  has no smooth transversal in general. We will generalise this monomorphism part to the case of inclusions of maximal compact subgroups  $K$  of semisimple Lie groups  $G$ . The geometry of  $G/K$  enters into this theorem via a Dirac operator  $\mathcal{D}_{G,K}$ . This generalisation (Theorem 9.1) is one of the key steps in a ‘quantisation commutes with *induction*’ result (Theorem 14.5) that we will use to deduce Theorem 6.13 from the compact case.

In the proof of Theorem 9.1, we will actually use the epimorphism case of naturality of the assembly map, Theorem 7.1, and Corollary 8.11 from the previous section, in Sections 9.1 and 9.4, respectively.

Let  $G$  be a connected semisimple Lie group with finite centre, and let  $K < G$  be a maximal compact subgroup. Let  $N$  be a smooth manifold,<sup>1</sup> equipped with a  $K$ -action. Let  $M := G \times_K N$  be the quotient of  $G \times N$  by the  $K$ -action given by

$$k \cdot (g, n) = (gk^{-1}, kn),$$

for  $k \in K$ ,  $g \in G$  and  $n \in N$ . Because this action is proper and free,  $M$  is a smooth manifold. Left multiplication on the factor  $G$  induces an action of  $G$  on  $M$ .

**Theorem 9.1** (Naturality of the assembly map for  $K \hookrightarrow G$ ). *The map  $\mathrm{K}\text{-Ind}_K^G$ , defined by commutativity of the left hand side of diagram (9.2), makes the following diagram commutative:*

$$\begin{array}{ccc} K_0^G(M) & \xrightarrow{\mu_M^G} & K_0(C_r^*(G)) \\ \mathrm{K}\text{-Ind}_K^G \uparrow & & \uparrow \mathrm{D}\text{-Ind}_K^G \\ K_0^K(N) & \xrightarrow{\mu_N^K} & R(K). \end{array} \tag{9.1}$$

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<sup>1</sup>In the previous two chapters, we used  $N$  to denote a normal subgroup. We hope this is not too confusing.

This result is analogous to Theorem 4.1 from [4], which is used by Paradan in [63] to reduce the Guillemin–Sternberg conjecture for compact groups to certain subgroups. Our proof of Theorem 6.13 is analogous to this part of Paradan’s work.

We will prove Theorem 9.1 by decomposing diagram (9.1) as follows:

$$\begin{array}{ccc}
 K_0^G(M) & \xrightarrow{\mu_M^G} & K_0(C_r^*(G)) \\
 \uparrow V_{\Delta(K)} & & \uparrow R_K^0 \\
 K_0^{G \times \Delta(K)}(G \times N) & \xrightarrow{\mu_{G \times N}^{G \times \Delta(K)}} & K_0(C_r^*(G \times K)) \\
 \uparrow \text{Res}_{G \times \Delta(K)}^{G \times K \times K} & & \uparrow \text{Res}_{G \times \Delta(K)}^{G \times K \times K} \\
 K_0^{G \times K \times K}(G \times N) & \xrightarrow{\mu_{G \times N}^{G \times K \times K}} & K_0(C_r^*(G \times K \times K)) \\
 \uparrow [\mathcal{D}_{G,K}] \times - & \mu_{G \times K}^{G \times K} [\mathcal{D}_{G,K}] \times - & \uparrow \\
 K_0^K(N) & \xrightarrow{\mu_N^K} & R(K).
 \end{array}
 \tag{9.2}$$

In this diagram, all the horizontal maps involving the letter  $\mu$  are analytic assembly maps. The symbol ‘ $\times$ ’ denotes the Kasparov product, and  $\Delta(K)$  is the diagonal subgroup of  $K \times K$ . The map  $\text{D-Ind}_K^G$  was defined in (6.11). The other maps will be defined in the remainder of this chapter.

The  $K$ -homology class  $[\mathcal{D}_{G,K}] \in K_0^{G \times K}(G)$  is defined as follows. Note that the Spin-Dirac operator on  $G/K$  is the operator  $\mathcal{D}_{G/K} = \mathcal{D}^{\mathbb{C}}$ , with  $\mathbb{C}$  the trivial  $K$ -representation, and  $\mathcal{D}^{\mathbb{C}}$  as in (6.9). Let  $p_G : G \rightarrow G/K$  be the quotient map, let  $\mathcal{S}^{G/K} := G \times_K \Delta_{\mathfrak{p}}$  be the spinor bundle on  $G/K$ , and consider the trivial vector bundle  $p_G^* \mathcal{S}^{G/K} = G \times \Delta_{\mathfrak{p}} \rightarrow G$ . Let  $\mathcal{D}_{G,K}$  be the operator on this bundle given by the same formula (6.9) as the operator  $\mathcal{D}^V$ , with  $V = \mathbb{C}$  the trivial representation. This operator satisfies

$$\mathcal{D}_{G,K}(p_G^* s) = p_G^*(\mathcal{D}^{\mathbb{C}} s),$$

for all sections  $s$  of  $\mathcal{S}^{G/K} \rightarrow G/K$ . We will use the fact that it is equivariant with respect to the action of  $G \times K$  on  $G \times \Delta_{\mathfrak{p}}$  defined by

$$(g, k) \cdot (g', \delta) = (gg'k^{-1}, \widetilde{\text{Ad}}(k) \cdot \delta),$$

for  $g, g' \in G$ ,  $k \in K$  and  $\delta \in \Delta_{\mathfrak{p}}$ . It is elliptic (see Lemma 15.6), and therefore defines a class  $[\mathcal{D}_{G,K}] \in K_0^{G \times K}(G)$ .

We will distinguish between the different subdiagrams of (9.2) by calling them the ‘left-hand’, ‘top’, ‘middle’, ‘bottom’ and ‘right-hand’ diagrams. Commutativity of the left-hand diagram is the definition of the map  $\text{K-Ind}_K^G$ . In this chapter we will prove that the other diagrams commute as well, thus giving a proof of Theorem 9.1.

## 9.1 The top diagram: naturality of the assembly map for epimorphisms

In this section, we suppose that  $G$  is a locally compact Hausdorff group, and that  $K \triangleleft G$  is a compact *normal* subgroup of  $G$ . Furthermore, let  $X$  be a locally compact, Hausdorff, proper  $G$ -space such that  $X/G$  is compact. Commutativity of the the top diagram is a special case of commutativity of the following diagram:

$$\begin{array}{ccccc}
 & & \xrightarrow{\mu_{X/K}^{G/K}} & & \\
 K_0^{G/K}(X/K) & \xrightarrow{\mu_{X/K}^{G/K}} & K_0(C^*(G/K)) & \xrightarrow{\lambda_{G/K}} & K_0(C_r^*(G/K)) \\
 \uparrow V_K & & \uparrow R_K^0 & & \uparrow R_K^0 \\
 K_0^G(X) & \xrightarrow{\mu_X^G} & K_0(C^*(G)) & \xrightarrow{\lambda_G} & K_0(C_r^*(G)) \\
 & & \xleftarrow{\mu_X^G} & & 
 \end{array} \tag{9.3}$$

We have used the same notation for the assembly map with respect to the full group  $C^*$ -algebra as for the assembly map with respect to the reduced one. The maps  $\lambda_{G/K}$  and  $\lambda_G$  were defined in Remark 5.16, where it was also noted that they make the top and bottom parts of diagram (9.3) commutative. The maps  $V_K$  and  $R_K^0$  are defined as in the epimorphism case of naturality of the assembly map, Theorem 7.1. It is a striking feature of our version of naturality of the assembly map for the monomorphism  $K \hookrightarrow G$  that it actually relies on the epimorphism case in this way.

It remains to prove that the right-hand part of diagram (9.3) commutes. But this is easily seen to be true, as the  $C^*$ -algebra homomorphisms that induce the maps  $R_K^0$ ,  $\lambda_G$  and  $\lambda_{G/K}$  commute on the dense subspace  $C_c(G)$  of  $C^*(G)$  (since the maps inducing  $\lambda_G$  and  $\lambda_{G/K}$  are the identity on  $C_c(G)$  and  $C_c(G/K)$ , respectively, and they are continuous).

## 9.2 The middle diagram: restriction to subgroups

In the middle diagram of (9.2), the map

$$\text{Res}_{G \times \Delta(K)}^{G \times K \times K} : K_0^{G \times K \times K}(G \times N) \rightarrow K_0^{G \times \Delta(K)}(G \times N)$$

is simply given by restricting representations and actions of  $G \times K \times K$  to  $G \times \Delta(K)$ . The other restriction map,

$$\text{Res}_{G \times \Delta(K)}^{G \times K \times K} : K_0(C_r^*(G \times K \times K)) \rightarrow K_0(C_r^*(G \times \Delta(K))), \tag{9.4}$$

is harder to define. (The restriction map  $C_c(G \times K \times K) \rightarrow C_c(G \times \Delta(K))$  is not continuous in the norms of the reduced group  $C^*$ -algebras involved, for example.)

We define the map (9.4) using the *K nneth formula*. Since  $G$  is a connected Lie group (in particular, it is an almost connected locally compact topological group), it satisfies the Baum–Connes conjecture with arbitrary  $G$ -trivial coefficients (see [16], Corollary 0.5). By Corollary

0.2 of [16], the algebra  $C_r^*(G)$  therefore satisfies the Künneth formula. In particular,

$$\begin{aligned} K_0(C_r^*(G \times K \times K)) &\cong K_0(C_r^*(G) \otimes_{\min} C_r^*(K \times K)) \\ &\cong K_0(C_r^*(G)) \otimes K_0(C_r^*(K \times K)) \\ &\cong K_0(C_r^*(G)) \otimes R(K \times K). \end{aligned}$$

Here we have used the fact that the representation ring  $R(K \times K)$  is torsion-free, and the fact that  $C_r^*(G_1) \otimes_{\min} C_r^*(G_2) \cong C_r^*(G_1 \otimes G_2)$  for all locally compact Hausdorff groups  $G_1$  and  $G_2$ . Analogously, we have an isomorphism  $K_0(C_r^*(G \times K)) \cong K_0(C_r^*(G)) \otimes R(K)$ .

The isomorphism is given by the Kasparov product. This product is defined as the composition

$$\begin{aligned} KK_0(\mathbb{C}, C_r^*(G)) \otimes KK_0(\mathbb{C}, C_r^*(K \times K)) &\xrightarrow{1 \otimes \tau_{C_r^*(G)}} \\ &KK_0(\mathbb{C}, C_r^*(G)) \otimes KK_0(C_r^*(G), C_r^*(G) \otimes_{\min} C_r^*(K \times K)) \xrightarrow{\times_{C_r^*(G)}} \\ &KK_0(\mathbb{C}, C_r^*(G) \otimes_{\min} C_r^*(K \times K)), \end{aligned} \quad (9.5)$$

where  $\tau_{C_r^*(G)}$  is defined by tensoring from the left by  $C_r^*(G)$ , and  $\times$  denotes the Kasparov product (see [10], Chapter 18.9). Let

$$\text{Res}_{\Delta(K)}^{K \times K} : R(K \times K) \rightarrow R(\Delta(K)) = R(K)$$

be the usual restriction map to the diagonal subgroup. We define (9.4) as the map

$$1_{K_0(C_r^*(G))} \otimes \text{Res}_{\Delta(K)}^{K \times K} : K_0(C_r^*(G)) \otimes R(K \times K) \rightarrow K_0(C_r^*(G)) \otimes R(K).$$

Commutativity of the middle diagram now follows from

**Lemma 9.2.** *Let  $X$  be a locally compact, Hausdorff, proper  $G \times K$ -space with compact quotient, and let  $Y$  be a compact, Hausdorff  $K$ -space. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0^{G \times \Delta(K)}(X \times Y) & \xrightarrow{\mu_{X \times Y}^{G \times \Delta(K)}} & K_0(C_r^*(G \times K)) \\ \text{Res}_{G \times \Delta(K)}^{G \times K \times K} \uparrow & & \uparrow \text{Res}_{G \times \Delta(K)}^{G \times K \times K} \\ K_0^{G \times K \times K}(X \times Y) & \xrightarrow{\mu_{X \times Y}^{G \times K \times K}} & K_0(C_r^*(G \times K \times K)). \end{array}$$

*Proof.* Let  $a = [\mathcal{H}, F, \pi] \in K_0^{G \times K \times K}(X \times Y)$ ,  $b = [\mathcal{E}_G, F_G] \in K_0(C_r^*(G))$  and  $[V] \in R(K \times K)$  be given, such that

$$\mu_{X \times Y}^{G \times K \times K}(a) = b \times [C_r^*(G) \otimes V] = [\mathcal{E}_G \otimes V, F_G \otimes 1_V] \in K_0(C_r^*(G \times K \times K)).$$

Because the assembly and restriction maps are  $\mathbb{Z}$ -module homomorphisms, it is sufficient to prove the claim in this case where the image of  $a$  is a simple tensor.

If we write

$$\begin{aligned} [\mathcal{E}, F_{\mathcal{E}}] &:= \mu_{X \times Y}^{G \times K \times K}(a) \in K_0(C_r^*(G \times K \times K)); \\ [\mathcal{E}', F_{\mathcal{E}'}] &:= \mu_{X \times Y}^{G \times \Delta(K)} \circ \text{Res}_{G \times \Delta(K)}^{G \times K \times K}(a) \in K_0(C_r^*(G \times K)), \end{aligned}$$

then the operators  $F_{\mathcal{E}}$  and  $F_{\mathcal{E}'}$  coincide on the dense mutual subspace  $\mathcal{H}_c$  of  $\mathcal{E}$  and  $\mathcal{E}'$ . It is therefore enough to prove that

$$\mathcal{E}' \cong \mathcal{E}_G \otimes_{\mathbb{C}} (V|_{\Delta(K)})$$

as Hilbert  $C_r^*(G \times K)$ -modules.

Using the usual choice of representatives of the classes  $b$  and  $[\mathcal{E}, F_{\mathcal{E}}]$  we have an isomorphism of Hilbert  $C_r^*(G \times K \times K)$ -modules

$$\psi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}_G \otimes V.$$

Define the map

$$\varphi : \mathcal{E}' \xrightarrow{\cong} \mathcal{E}_G \otimes (V|_{\Delta(K)})$$

by  $\varphi|_{\mathcal{H}_c} = \psi|_{\mathcal{H}_c}$ , and continuous extension. The map  $\varphi$  is well-defined, and indeed an isomorphism, if it is a homomorphism of Hilbert  $C_r^*(G \times K)$ -modules. To show that  $\varphi$  preserves the  $C_r^*(G \times K)$ -valued inner products, let  $\xi_1, \xi_2 \in \mathcal{H}_c$  be given, and suppose that  $\varphi(\xi_j) = e_j \otimes v_j \in \mathcal{E}_G \otimes V$  for  $j = 1, 2$ . (By linearity of  $\varphi$ , it is indeed enough to consider the case where the  $\varphi(\xi_j)$  are simple tensors.) Then for all  $g \in G$  and  $k \in K$ ,

$$\begin{aligned} (\varphi(\xi_1), \varphi(\xi_2))_{\mathcal{E}_G \otimes V|_{\Delta(K)}}(g, k) &= (e_1, e_2)_{\mathcal{E}_G}(g) (v_1, (k, k) \cdot v_2)_V \\ &= (\psi(\xi_1), \psi(\xi_2))_{\mathcal{E}_G \otimes V}(g, k, k) \\ &= (\xi_1, \xi_2)_{\mathcal{E}}(g, k, k), \end{aligned}$$

because  $\psi$  is an isomorphism of Hilbert  $C^*(G \times K \times K)$ -modules. The latter expression equals

$$(\xi_1, (g, k, k) \cdot \xi_2)_{\mathcal{H}} = (\xi_1, \xi_2)_{\mathcal{E}'}(g, k),$$

which shows that  $\varphi$  preserves the inner products.

Finally, because  $\psi$  is a homomorphism of  $C_r^*(G \times K \times K)$ -modules, the map  $\varphi$  is a homomorphism of  $C_r^*(G \times K)$ -modules on  $\mathcal{H}_c$ , and hence on all of  $\mathcal{E}'$ .  $\square$

### 9.3 The bottom diagram: multiplicativity of the assembly map

Commutativity of the bottom diagram is a special case of the multiplicativity property of the assembly map that we will prove in this section. This property generalises multiplicativity of the index with respect to Atiyah's 'sharp product' of elliptic operators, as described in [4], Theorem 3.5. In this section, we will denote the tensor product of Hilbert  $C^*$ -modules (see Definition 5.1) by  $\hat{\otimes}$ , to emphasise the difference with the algebraic tensor product  $\otimes$ .

For this section, let  $G_1$  and  $G_2$  be locally compact Hausdorff topological groups, acting properly on two locally compact metrisable spaces  $X_1$  and  $X_2$ , respectively. Suppose  $X_1/G_1$  and  $X_2/G_2$  are compact. Consider the Kasparov product maps

$$\begin{aligned} K_0^{G_1}(X_1) \otimes K_0^{G_2}(X_2) &\xrightarrow{\times} K_0^{G_1 \times G_2}(X_1 \times X_2); \\ K_0(C_{(r)}^*(G_1)) \otimes K_0(C_{(r)}^*(G_2)) &\xrightarrow{\times} K_0(C_{(r)}^*(G_1 \times G_2)). \end{aligned} \tag{9.6}$$

Here the symbol  $C_{(r)}^*$  denotes either the full or the reduced group  $C^*$ -algebra, and we have used the  $C^*$ -algebra isomorphisms (4.8) and (4.9).

Analogously to (9.5), the Kasparov product (9.6) is actually the composition

$$\begin{aligned} KK_0(\mathbb{C}, C_{(r)}^*(G_1)) \otimes KK_0(\mathbb{C}, C_{(r)}^*(G_2)) &\xrightarrow{1 \otimes \tau_{C_{(r)}^*(G_1)}} \\ &KK_0(\mathbb{C}, C_{(r)}^*(G_1)) \otimes KK_0(C_{(r)}^*(G_1), C_{(r)}^*(G_1) \otimes C_{(r)}^*(G_2)) \xrightarrow{\times_{C_{(r)}^*(G_1)}} \\ &KK_0(\mathbb{C}, C_{(r)}^*(G_1) \otimes C_{(r)}^*(G_2)) = KK_0(\mathbb{C}, C_{(r)}^*(G_1 \times G_2)). \end{aligned} \quad (9.7)$$

The tensor product denotes the maximal tensor product in the case of full  $C^*$ -algebras, and the minimal tensor product for reduced  $C^*$ -algebras.

**Theorem 9.3** (Multiplicativity of the assembly map). *If  $X_1$  and  $X_2$  are metrisable, then for all  $a_j \in K_0^{G_j}(X_j)$ , we have*

$$\mu_{X_1}^{G_1}(a_1) \times \mu_{X_2}^{G_2}(a_2) = \mu_{X_1 \times X_2}^{G_1 \times G_2}(a_1 \times a_2) \in K_0(C_{(r)}^*(G_1 \times G_2)).$$

Here the assembly maps are defined with respect to either the full or the reduced group  $C^*$ -algebras. We suppose  $X_1$  and  $X_2$  to be metrisable, because the  $C^*$ -algebras  $C_0(X_1)$  and  $C_0(X_2)$  are then separable, so that we can use Baaj and Julg's unbounded description of the Kasparov product. Theorem 9.3 may well be true for non-metrisable spaces, but we will only apply it to smooth manifolds anyway.

In the proof of Theorem 9.3, we will use the unbounded picture of  $KK$ -theory (see Section 5.3), because of the easy form of the Kasparov product in this setting. The construction of the unbounded assembly map in Section 5.3 works for full group  $C^*$ -algebras, so the following proof applies only to this case. Theorem 9.3 for reduced group  $C^*$ -algebras can then be deduced using the maps  $\lambda_{G_1}$  and  $\lambda_{G_2}$  defined in Remark 5.16.

*Proof of Theorem 9.3.* For  $j = 1, 2$ , let

$$a_j = (\mathcal{H}_j, D_j, \pi_j) \in \Psi_0^{G_j}(C_0(X_j), \mathbb{C})$$

be given. Then

$$\tilde{\mu}_{X_j}^{G_j}(a_j) = (\tilde{\mathcal{E}}_j, D_{\tilde{\mathcal{E}}_j}),$$

as in (5.11). The product of  $\tilde{\mu}_{X_1}^{G_1}(a_1)$  and  $\tilde{\mu}_{X_2}^{G_2}(a_2)$  is

$$\tilde{\mu}_{X_1}^{G_1}(a_1) \times \tilde{\mu}_{X_2}^{G_2}(a_2) = (\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2, D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}) \in \Psi_0(\mathbb{C}, C^*(G_1 \times G_2)). \quad (9.8)$$

Here  $D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$  is the closure of the operator

$$D_{\tilde{\mathcal{E}}_1} \otimes 1_{\tilde{\mathcal{E}}_2} + 1_{\tilde{\mathcal{E}}_1} \otimes D_{\tilde{\mathcal{E}}_2},$$

on the domain  $\text{dom } D_{\tilde{\mathcal{E}}_1} \otimes \text{dom } D_{\tilde{\mathcal{E}}_2}$ .

On the other hand, the product  $a_1 \times a_2$  is

$$(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, D_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}, \pi) \in \Psi_0^{G_1 \times G_2}(C_0(X_1 \times X_2), \mathbb{C}), \quad (9.9)$$

with  $D_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$  the closure of the operator

$$D_1 \otimes 1_{\mathcal{H}_2} + 1_{\mathcal{H}_1} \otimes D_2$$

on  $\text{dom } D_1 \otimes \text{dom } D_2$ . Furthermore, we have abbreviated  $\pi := \pi_1 \otimes \pi_2$  for later convenience. Applying the unbounded assembly map  $\tilde{\mu}_{X_1 \times X_2}^{G_1 \times G_2}$  to the cycle (9.9), we obtain

$$(\tilde{\mathcal{E}}, D_{\tilde{\mathcal{E}}}) \in \Psi_0(\mathbb{C}, C^*(G_1 \times G_2)), \quad (9.10)$$

where  $\tilde{\mathcal{E}} := \widetilde{\tilde{\pi}(p) \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$ . Here  $p := p_1 \otimes p_2$ , with  $p_j$  the projection in  $C_c(X_j \times G_j)$  as defined in (5.9). Furthermore, the operator  $D_{\tilde{\mathcal{E}}}$  is the closure of the operator  $\tilde{D}_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_1}$ , as defined in (5.10), with  $D = D_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$ .

First, let us show that  $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2$ . Note that  $\widetilde{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$  is the completion of the space  $C_c(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$  with respect to the  $C^*(G_1 \times G_2)$ -valued inner product  $(-, -)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$ , defined analogously to (5.8). On the other hand,

$$\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2 = \tilde{\pi}_1(p_1) \tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\pi}_2(p_2) \tilde{\mathcal{H}}_2 = \tilde{\pi}(p) \tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2,$$

since it is not hard to check that  $\tilde{\pi}(f_1 \otimes f_2) = \tilde{\pi}_1(f_1) \otimes \tilde{\pi}_2(f_2)$  for all  $f_j \in C_c(X_j \times G_j)$ . Here  $\tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2$  is the completion of  $C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2)$  in the  $C^*(G_1) \otimes C^*(G_2) \cong C^*(G_1 \times G_2)$ -valued inner product given by

$$(\varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} = (\varphi_1, \psi_1)_{\mathcal{H}_1} \otimes (\varphi_2, \psi_2)_{\mathcal{H}_2},$$

for  $\varphi_j, \psi_j \in C_c(G_j, \mathcal{H}_j)$ . It follows directly from the definition (5.8) of the inner products  $(-, -)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$  and  $(-, -)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$ , that they coincide on the subspace  $C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2) \subset C_c(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ .

We claim that the completion of  $C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2)$  with respect to this inner product contains the space  $C_c(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ . Then we indeed have  $\widetilde{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} \cong \tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2$ , and hence

$$\tilde{\mathcal{E}} = \tilde{\pi}(p) (\widetilde{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}) \cong \tilde{\pi}(p) (\tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2) = \tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2,$$

as Hilbert  $C^*(G_1 \times G_2)$ -modules. The proof of this claim is based on the inequality

$$\begin{aligned} \|(\varphi, \varphi)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}\|_{C^*(G_1 \times G_2)} &\leq \|\varphi\|_{L^1(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)}^2 \\ &:= \left( \int_{G_1 \times G_2} \|\varphi(g_1, g_2)\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} dg_1 dg_2 \right)^2, \end{aligned} \quad (9.11)$$

for all  $\varphi \in C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2)$ . This inequality is proved in Lemma 9.4 below. Because of this estimate, the completion of  $C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2)$  with respect to the inner product  $(-, -)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}$  contains the completion of this tensor product in the norm  $\|\cdot\|_{L^1(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)}$ , which in turn contains  $C_c(G_1 \times G_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ .

Next, we prove that the two unbounded cycles (9.8) and (9.10) define the same class in  $KK$ -theory. By Lemma 10 and Corollary 17 from [47], this follows if we can show that

$$\text{dom } D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2} \subset \text{dom } D_{\tilde{\mathcal{E}}}, \quad \text{and} \quad (9.12)$$

$$D_{\tilde{\mathcal{E}}} \big|_{\text{dom } D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}} = D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}. \quad (9.13)$$



We first prove (9.12). Note that the domain of  $D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$  is the completion of  $\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$ , as in (4.15), given by

$$\|\varphi_1 \otimes \varphi_2\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}^2 := \|\varphi_1 \otimes \varphi_2\|_{\tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2}^2 + \|D_{\tilde{\mathcal{E}}_1} \varphi_1 \otimes \varphi_2 + \varphi_1 \otimes D_{\tilde{\mathcal{E}}_2} \varphi_2\|_{\tilde{\mathcal{H}}_1 \hat{\otimes} \tilde{\mathcal{H}}_2}^2, \quad (9.14)$$

for all  $\varphi_j \in \text{dom} D_{\tilde{\mathcal{E}}_j}$ . The domain of  $D_{\tilde{\mathcal{E}}_j}$  in turn is the completion of  $\tilde{\pi}_j(p_j)C_c(G_j, \text{dom} D_j)$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_j}}$ , defined analogously to (9.14).

To prove (9.12), we consider the subspace

$$V := \tilde{\pi}_1(p_1)C_c(G_1, \text{dom} D_1) \otimes \tilde{\pi}_2(p_2)C_c(G_2, \text{dom} D_2)$$

of  $\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}$ . We begin by showing that the completion of  $V$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$  contains  $\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}$ . This will imply that

$$\begin{aligned} \overline{V} &= \overline{\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}} \\ &= \text{dom} D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}, \end{aligned} \quad (9.15)$$

with completions taken in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$ .

For  $j = 1, 2$ , let  $\varphi_j \in \text{dom} D_{\tilde{\mathcal{E}}_j}$  be given. Let  $(\varphi_j^k)_{k=1}^\infty$  be a sequence in  $\tilde{\pi}_j(p_j)C_c(G_j, \text{dom} D_j)$  such that

$$\lim_{k \rightarrow \infty} \|\varphi_j^k - \varphi_j\|_{D_{\tilde{\mathcal{E}}_j}} = 0.$$

We claim that

$$\lim_{k \rightarrow \infty} \|\varphi_1^k \otimes \varphi_2^k - \varphi_1 \otimes \varphi_2\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}} = 0, \quad (9.16)$$

which implies that  $\varphi_1 \otimes \varphi_2$  lies in the completion of  $V$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$ . This claim is proved in Lemma 9.5 below. General elements of  $\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}$  are (finite) sums of simple tensors like  $\varphi_1 \otimes \varphi_2$ , and can be approximated by sums of sequences like  $(\varphi_1^k \otimes \varphi_2^k)_{k=1}^\infty$ . Hence the completion of  $V$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$  indeed contains  $\text{dom} D_{\tilde{\mathcal{E}}_1} \otimes \text{dom} D_{\tilde{\mathcal{E}}_2}$ , so that (9.15) holds.

Finally, observe that  $\text{dom} D_{\tilde{\mathcal{E}}}$  is the completion of

$$\pi(p)C_c(G_1 \times G_2, \text{dom} D_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2})$$

in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}}}$ , which is again defined analogously to (9.14). Since  $V$  is contained in  $\pi(p)C_c(G_1 \times G_2, \text{dom} D_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2})$ , the completion of  $V$  in the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}}}$  is contained in  $\text{dom} D_{\tilde{\mathcal{E}}}$ . Furthermore, the operators  $D_{\tilde{\mathcal{E}}}$  and  $D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$  coincide on  $V$ , since their restrictions to  $V$  are both given by

$$\tilde{\pi}_1(p_1)\varphi_1 \otimes \tilde{\pi}_2(p_2)\varphi_2 \mapsto \tilde{\pi}_1(p_1)D_1 \circ \varphi_1 \otimes \tilde{\pi}_2(p_2)\varphi_2 + \tilde{\pi}_1(p_1)\varphi_1 \otimes \tilde{\pi}_2(p_2)D_2 \circ \varphi_2.$$

This implies that the norms  $\|\cdot\|_{D_{\tilde{\mathcal{E}}}}$  and  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$  are the same on  $V$ , so that the completion of  $V$  with respect to  $\|\cdot\|_{D_{\tilde{\mathcal{E}}}}$  equals the completion of  $V$  with respect to  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$ , which equals  $\text{dom} D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$ , by (9.15). We conclude that

$$\text{dom} D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2} = \overline{V} \subset \text{dom} D_{\tilde{\mathcal{E}}},$$

as claimed.

Claim (9.13) now follows, because by (9.15), the restriction of  $D_{\tilde{\mathcal{E}}}$  to  $\text{dom } D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$  is the closure of  $D_{\tilde{\mathcal{E}}}|_V$ , which equals  $D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}|_V$ . The closure of the latter operator is  $D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}$ , again by (9.15), and we are done.  $\square$

**Lemma 9.4.** *The inequality (9.11) holds for all  $\varphi \in C_c(G_1, \mathcal{H}_1) \otimes C_c(G_2, \mathcal{H}_2)$ .*

*Proof.* For such  $\varphi$ , we have

$$\begin{aligned} \|(\varphi, \varphi)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}\|_{C^*(G_1 \times G_2)} &\leq \|(\varphi, \varphi)_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}\|_{L^1(G_1 \times G_2)} \\ &= \int_{G_1 \times G_2} \left| \int_{G_1 \times G_2} (\varphi(g'_1, g'_2), \varphi(g'_1 g_1, g'_2 g_2))_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} dg'_1 dg'_2 \right| dg_1 dg_2 \\ &\leq \int_{G_1 \times G_2} \int_{G_1 \times G_2} \left| (\varphi(g'_1, g'_2), \varphi(g'_1 g_1, g'_2 g_2))_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} \right| dg'_1 dg'_2 dg_1 dg_2 \\ &\leq \int_{G_1 \times G_2} \int_{G_1 \times G_2} \|\varphi(g'_1, g'_2)\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} \|\varphi(g'_1 g_1, g'_2 g_2)\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} dg'_1 dg'_2 dg_1 dg_2, \end{aligned}$$

by the Cauchy-Schwartz inequality. Because of left invariance of the Haar measures  $dg_1$  and  $dg_2$ , the latter expression is the square of the  $L^1$ -norm of  $\varphi$ .  $\square$

**Lemma 9.5.** *The limit (9.16) equals zero.*

*Proof.* Since for  $j = 1, 2$ , we have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \|\varphi_j^k - \varphi_j\|_{D_{\tilde{\mathcal{E}}_j}}^2 \\ &= \lim_{k \rightarrow \infty} \left( \|\varphi_j^k - \varphi_j\|_{\mathcal{H}_j}^2 + \|D_{\tilde{\mathcal{E}}_j} \varphi_j^k - D_{\tilde{\mathcal{E}}_j} \varphi_j\|_{\mathcal{H}_j}^2 \right), \end{aligned} \quad (9.17)$$

both terms in (9.17) tend to zero as  $k \rightarrow \infty$ . Let us rewrite (9.16) in a way that allows us to use this fact. By definition of the norm  $\|\cdot\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}$ , we have

$$\begin{aligned} \|\varphi_1^k \otimes \varphi_2^k - \varphi_1 \otimes \varphi_2\|_{D_{\tilde{\mathcal{E}}_1 \hat{\otimes} \tilde{\mathcal{E}}_2}}^2 &= \\ &= \|\varphi_1^k \otimes \varphi_2^k - \varphi_1 \otimes \varphi_2\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}^2 + \\ &+ \|D_{\tilde{\mathcal{E}}_1} \varphi_1^k \otimes \varphi_2^k - D_{\tilde{\mathcal{E}}_1} \varphi_1 \otimes \varphi_2 + \tilde{\varphi}_1^k \otimes D_{\tilde{\mathcal{E}}_2} \varphi_2^k - \varphi_1 \otimes D_{\tilde{\mathcal{E}}_2} \varphi_2\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2}^2. \end{aligned}$$

Using the triangle inequality and the fact that

$$\|\psi_1 \otimes \psi_2\|_{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2} \leq \|\psi_1\|_{\mathcal{H}_1} \|\psi_2\|_{\mathcal{H}_2}$$

for all  $\psi_j \in \mathcal{H}_j$  (this follows from the fact that any  $C^*$ -norm on a tensor product is *subcross*, see [87], Corollary T.6.2), we see that this number is less than or equal to

$$\begin{aligned} &\left( \|\varphi_1^k - \varphi_1\|_{\mathcal{H}_1} \|\varphi_2^k\|_{\mathcal{H}_2} + \|\varphi_1\|_{\mathcal{H}_1} \|\varphi_2^k - \varphi_2\|_{\mathcal{H}_2} \right)^2 + \\ &\left( \|D_{\tilde{\mathcal{E}}_1} \varphi_1^k - D_{\tilde{\mathcal{E}}_1} \varphi_1\|_{\mathcal{H}_1} \|\varphi_2^k\|_{\mathcal{H}_2} + \|D_{\tilde{\mathcal{E}}_1} \varphi_1\|_{\mathcal{H}_1} \|\varphi_2^k - \varphi_2\|_{\mathcal{H}_2} + \right. \\ &\quad \left. \|\varphi_1^k - \varphi_1\|_{\mathcal{H}_1} \|D_{\tilde{\mathcal{E}}_2} \varphi_2^k\|_{\mathcal{H}_2} + \|\varphi_1\|_{\mathcal{H}_1} \|D_{\tilde{\mathcal{E}}_2} \varphi_2^k - D_{\tilde{\mathcal{E}}_2} \varphi_2\|_{\mathcal{H}_2} \right)^2. \end{aligned} \quad (9.18)$$

By the observation at the beginning of this proof, all terms in (9.18) contain a factor that goes to zero as  $k \rightarrow \infty$ . Since the other factors are bounded functions of  $k$ , the claim follows.  $\square$

## 9.4 The right-hand diagram: a decomposition of the induction map $\text{D-Ind}_K^G$

In this section, we complete the proof of Theorem 9.1 by proving commutativity of the right-hand diagram in (9.2). In this proof, we will use commutativity of the top, middle and bottom diagrams in the case where  $N$  is a point.

But first, we give the following description of the map  $\text{D-Ind}_K^G$ . Let  $V$  be a finite-dimensional unitary representation of  $K$ , and let  $\mathcal{D}^V$  be the Dirac operator defined in (6.9). The closure of this operator is an unbounded self-adjoint operator on the space of  $L^2$ -sections of  $E_V$ , which is odd with respect to the  $\mathbb{Z}_2$ -grading. This space of  $L^2$ -sections is isomorphic to the space  $(L^2(G) \otimes \Delta_{d_p} \otimes V)^K$ , where the  $K$ -action is again defined by (6.7) (with smooth functions replaced by  $L^2$ -functions, of course). Let  $b$  be a normalising function, so that we have the class

$$[(L^2(G) \otimes \Delta_{d_p} \otimes V)^K, b(\mathcal{D}^V), \pi_{G/K}] \in K_0^G(G/K).$$

Here  $\pi_{G/K}$  denotes the representation of  $C_0(G/K)$  on  $L^2(G/K, E_V)$  as multiplication operators.

**Lemma 9.6.** *In this situation, we have*

$$\text{D-Ind}_K^G[V] = \mu_{G/K}^G[(L^2(G) \otimes \Delta_{d_p} \otimes V)^K, b(\mathcal{D}^V), \pi_{G/K}] \in K_0(C_r^*(G)).$$

*Proof.* Write

$$[\mathcal{E}, F_{\mathcal{E}}] := \mu_{G/K}^G[(L^2(G) \otimes \Delta_{d_p} \otimes V)^K, b(\mathcal{D}^V), \pi_{G/K}].$$

Since the restriction of  $F_{\mathcal{E}}$  to  $(C_c(G) \otimes \Delta_{d_p} \otimes V)^K$  is the restriction of  $b(\mathcal{D}^V)$  to this space, we only need to prove that

$$\mathcal{E} = (C_r^*(G) \otimes \Delta_{d_p} \otimes V)^K \quad (9.19)$$

as Hilbert  $C_r^*(G)$ -modules.

To prove this equality, we note that for all  $f, f' \in (L^2(G))_c$  and all  $g \in G$ ,

$$(f, f')_{\mathcal{E}}(g) = (f, g \cdot f')_{L^2(G)} = (f * (f')^*)(g),$$

as one easily computes. This implies that the  $C_r^*G$ -valued inner product on  $\mathcal{E}$  is the same as the one on  $(C_r^*(G) \otimes \Delta_{d_p} \otimes V)^K$ .

The  $C_r^*(G)$ -module structure of  $\mathcal{E}$  is given by

$$\begin{aligned} h \cdot (f \otimes \delta \otimes v) &= \int_G h(g) g \cdot (f \otimes \delta \otimes v) dg \\ &= (h * f) \otimes \delta \otimes v, \end{aligned}$$

for all  $h \in C_c(G)$ ,  $f \in L^2(G)$ ,  $\delta \in \Delta_{d_p}$  and  $v \in V$ . Hence the equality (9.19) includes the  $C_r^*(G)$ -module structure.  $\square$

*Proof of commutativity of the right-hand diagram.* Consider the vector bundles  $V$  and  $\{0\}$  over a point. Let  $0_V : V \rightarrow \{0\}$  be the only possible operator between (the spaces of smooth sections of) these bundles. It defines a class  $[0_V] = [V \oplus \{0\}, 0_V] \in K_0^K(\text{pt})$ , and we have

$$\mu_{\text{pt}}^K[0_V] = [V] \in R(K).$$

Now we find that

$$\text{D-Ind}_K^G[V] = \mu_{G/K}^{G \times K \times K} [(L^2(G) \otimes \Delta_{d_p} \otimes V)^K, b(\mathcal{D}^V), \pi_{G/K}]$$

by Lemma 9.6,

$$= \mu_{G/K}^G \circ V_{\Delta(K)} \circ \text{Res}_{G \times \Delta(K)}^{G \times K \times K} [\mathcal{D}_{G,K} \otimes 1_V]$$

by Corollary 8.11 and the fact that  $\mathcal{D}^V$  is the restriction of  $\mathcal{D}_{G,K} \otimes 1_V$  to  $K$ -invariant elements of  $C^\infty(G) \otimes \Delta_{d_p} \otimes V$ ,

$$\begin{aligned} &= \mu_{G/K}^G \circ V_{\Delta(K)} \circ \text{Res}_{G \times \Delta(K)}^{G \times K \times K} ([\mathcal{D}_{G,K}] \times [0_V]) \\ &= R_K^0 \circ \text{Res}_{G \times \Delta(K)}^{G \times K \times K} \circ \mu_G^{G \times K} ([\mathcal{D}_{G,K}] \times [V]), \end{aligned}$$

by commutativity of the top, middle and bottom diagrams when  $N$  is a point.  $\square$

**Remark 9.7.** Supposing that  $V$  is irreducible, we could also have applied the Borel–Weil(–Bott) theorem to realise the class  $[V] \in R(K)$  as  $\mu_{K/T}^K [\mathcal{D}_{i\lambda}]$ , where  $i\lambda$  is the highest weight of  $V$ , and  $\mathcal{D}_{i\lambda}$  is the Dolbeault–Dirac operator on  $K/T$  coupled to the usual line bundle that is used in the Borel–Weil theorem. We would then have used commutativity of the top, middle and bottom diagrams for  $N = K/T$ .

## **Part III**

### **Groups with a cocompact, discrete, normal subgroup**

This part is devoted to a proof of Theorem 6.5. The ingredients of this proof are:

1. the Guillemin–Sternberg conjecture in the compact case (Theorem 3.34);
2. the epimorphism part of naturality of the assembly map (Theorem 7.1);
3. symplectic reduction in stages (Theorem 2.25);
4. quantum reduction in stages (10.5);
5. specialisation (10.8) of Corollary 8.11 to Dirac operators, in the case of a free action by a discrete group.

We combine these ingredients into Diagram 10.1, which gives an outline of our proof. The main technical step that then remains is Proposition 10.1, which we prove in Section 10.3.

In Chapter 11, we illustrate Theorem 6.5 by giving an independent proof of this theorem, in the case that  $G$  is discrete and abelian. This proof, based on a paper by Lusztig [55] (see also [8], pp. 242–243) gives considerable insight in the situation, and does not rely on naturality of the assembly map. It is based on an explicit computation of the image under  $\mu_M^\Gamma$  of a  $K$ -homology class  $[D]$  associated to a  $\Gamma$ -equivariant elliptic differential operator  $D$  on a  $\Gamma$ -vector bundle  $E$  over a  $\Gamma$ -manifold  $M$ . Because in this case  $C^*(\Gamma) \cong C(\hat{\Gamma})$  (with  $\hat{\Gamma}$  the unitary dual of  $\Gamma$ ), this image corresponds to the formal difference of two equivalence classes of vector bundles over  $\hat{\Gamma}$ . These bundles are described as the kernel and cokernel of a ‘field of operators’  $(D_\alpha)_{\alpha \in \hat{\Gamma}}$  on a ‘field of vector bundles’  $(E_\alpha \rightarrow M/\Gamma)_{\alpha \in \hat{\Gamma}}$ . The operators  $D_\alpha$  and the bundles  $E_\alpha$  are constructed explicitly from  $D$  and  $E$ , respectively. The quantum reduction of the class  $\mu_M^\Gamma[D]$  is the index of the operator  $D_1$  on  $E_1 \rightarrow M/\Gamma$ , where  $1 \in \hat{\Gamma}$  is the trivial representation. Because  $D_1$  is the operator  $D^\Gamma$  mentioned above, in this case Theorem 6.5 follows from the computation in Chapter 10.

Finally, in Sections 11.5 and 11.6 we check the discrete abelian case in an explicit computation. We will see that the quantisation of the action of  $\mathbb{Z}^2$  on  $\mathbb{R}^2$  corresponds to a certain line bundle over the two-torus  $\mathbb{T}^2 = \hat{\mathbb{Z}}^2$ . The quantum reduction of this  $K$ -theory class is the rank of this line bundle, the integer 1. This is also the quantisation of the reduced space  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ , as can be seen either directly or by applying Atiyah–Singer for Dirac operators. Although this is the simplest example of Guillemin–Sternberg for noncompact groups and spaces, it is not a trivial matter to find a suitable prequantisation in this case.

# Chapter 10

## Dirac operators and the map $V_\Gamma$

In this chapter, we finish the proof of Theorem 6.5. We first sketch an outline of this proof in Section 10.1, and then state and prove the remaining technical step in Sections 10.2 and 10.3.

### 10.1 Outline of the proof

We use the notation and assumptions from Section 6.1 and Theorem 6.5. In particular,  $G$  is a Lie group,  $\Gamma \triangleleft G$  is a discrete normal subgroup, such that  $K := G/\Gamma$  is compact. Furthermore,  $(M, \omega)$  is a proper Hamiltonian  $G$ -manifold, on which  $\Gamma$  acts freely. The assumption that  $M/G$  is compact is now equivalent to compactness of  $M/\Gamma$ .

The third and fourth ingredients mentioned at the beginning of the introduction to Part III allow us to set up the following diagram:

$$\begin{array}{ccccc}
 \text{Preq}(G \curvearrowright M, \omega) & \xrightarrow{[\bar{\partial}_\bullet + \bar{\partial}_\bullet^*]} & K_0^G(M) & \xrightarrow{\mu_M^G} & K_0(C^*(G)) \\
 \downarrow R_\Gamma^0 & & & & \downarrow R_\Gamma^0 \\
 \text{Preq}(K \curvearrowright M//\Gamma, \omega_{M//\Gamma}) & \xrightarrow{[\bar{\partial}_\bullet + \bar{\partial}_\bullet^*]} & K_0^K(M//\Gamma) & \xrightarrow{\mu_{M//\Gamma}^K} & K_0(C^*(K)) \\
 \downarrow R_K^0 & & & & \downarrow R_K^0 \\
 \text{Preq}((M//\Gamma)//K, \omega_{(M//\Gamma)//K}) & \xrightarrow{\text{index}(\bar{\partial}_\bullet + \bar{\partial}_\bullet^*)} & & & \mathbb{Z}.
 \end{array} \tag{10.1}$$

Here the following notation is used.  $\text{Preq}(G \curvearrowright M, \omega)$  is the set of all  $G$ -equivariant prequantisations of  $(M, \omega)$ . A necessary condition for  $\text{Preq}(G \curvearrowright M, \omega)$  to be nonempty is the requirement that the cohomology class  $[\omega] \in H^2(M, \mathbb{R})$  be integral. Since we assume  $(M, \omega)$  to be equivariantly prequantisable, this condition must be satisfied. Similarly,  $\text{Preq}(K \curvearrowright M//\Gamma, \omega_{M//\Gamma})$  is defined given the  $K$ -action on  $M//\Gamma$  induced by the  $G$ -action on  $M$ , and  $\text{Preq}((M//\Gamma)//K, \omega_{(M//\Gamma)//K})$  is just the set of prequantisations of the symplectic orbifold

$$((M//\Gamma)//K, \omega_{(M//\Gamma)//K}) \cong (M//G, \omega_{M//G}); \tag{10.2}$$

this isomorphism follows from Theorem 2.25. Note that in this case,  $M//\Gamma = M/\Gamma$ , since  $\Gamma$  is discrete.

The maps  $R_\Gamma^0$  and  $R_K^0$  on the left hand (classical) side of (10.1) are given by the construction (3.14) of a prequantisation on a symplectic reduction, induced by an equivariant prequantisation on the original manifold. The quantum counterparts of these maps on the right hand side of (10.1) are defined by

$$R_\Gamma^0 := (\Sigma_\Gamma)_*; \quad (10.3)$$

$$R_K^0 := (f_K)_*. \quad (10.4)$$

Here  $(\Sigma_\Gamma)_* : K_0(C^*(G)) \rightarrow K_0(C^*(K))$  is the map functorially induced by the map  $\Sigma_\Gamma : C^*(G) \rightarrow C^*(G/\Gamma)$  given by

$$(\Sigma_\Gamma f)(\Gamma g) = \sum_{\gamma \in \Gamma} f(\gamma g),$$

initially defined on  $f \in C_c(G)$  and continuously extended to all of  $C^*(G)$ . This map was more generally defined for any closed normal subgroup  $N$  of  $G$  in (7.1). Finally, the maps  $[\bar{\partial}_\bullet + \bar{\partial}_\bullet^*]$  are defined by taking the  $K$ -homology class of the Dolbeault–Dirac operator coupled to a given prequantum line bundle, as explained in Corollary 4.36. Thus the commutativity of the upper part of diagram (10.1) is the equality

$$\mu_{M/\Gamma}^K[\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*] = R_\Gamma^0(\mu_M^G[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*]),$$

for any prequantum line bundle  $L^\omega \rightarrow M$ . Commutativity of the lower part is the statement

$$\text{index}(\bar{\partial}_{L^\omega(M/\Gamma)/K} + \bar{\partial}_{L^\omega(M/\Gamma)/K}^*) = R_K^0(\mu_{M/\Gamma}^K[\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*]).$$

It is easily shown that

$$f_K \circ \Sigma_\Gamma = f_G,$$

so that by functoriality of  $K_0$ , one has

$$R_K^0 \circ R_\Gamma^0 = R_G^0. \quad (10.5)$$

The classical version of (10.5) follows from (10.2). Using the classical and quantum versions of this equality, we see that the outer diagram in (10.1) is equal to

$$\begin{array}{ccc} \text{Preq}(G \circlearrowleft M, \omega) & \xrightarrow{Q_V} & K_0(C^*(G)) \\ \downarrow R_G^0 & & \downarrow R_G^0 \\ \text{Preq}(M_G, \omega_G) & \xrightarrow{Q_V} & \mathbb{Z}. \end{array} \quad (10.6)$$

Here  $Q_V$  is the quantisation map of Definition 6.1, so that commutativity of diagram (10.6) is precisely Theorem 6.5.

We will prove commutativity of diagram (10.6) by showing that the two inner diagrams in (10.1) commute. Now the lower diagram commutes by the validity of the Guillemin–Sternberg conjecture for compact  $K$  (Theorem 3.34), whereas the upper diagram decomposes as

$$\begin{array}{ccccc} \text{Preq}(G \circlearrowleft M, \omega) & \xrightarrow{[\bar{\partial}_\bullet + \bar{\partial}_\bullet^*]} & K_0^G(M) & \xrightarrow{\mu_M^G} & K_0(C^*(G)) \\ \downarrow R_\Gamma^0 & & \downarrow V_\Gamma & & \downarrow R_\Gamma^0 \\ \text{Preq}(K \circlearrowleft M/\Gamma, \omega_{M/\Gamma}) & \xrightarrow{[\bar{\partial}_\bullet + \bar{\partial}_\bullet^*]} & K_0^K(M/\Gamma) & \xrightarrow{\mu_{M/\Gamma}^K} & K_0(C^*(K)), \end{array} \quad (10.7)$$



where  $V_\Gamma$  is the map defined in Section 7.4, with  $N = \Gamma$ . The right hand inner diagram in (10.7) commutes by the epimorphism case of naturality of the assembly map, Theorem 7.1. So it is only left to prove that the left hand diagram in (10.7) commutes. Explicitly, commutativity of this diagram means that

$$V_\Gamma[\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*] = [\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*]. \quad (10.8)$$

We will deduce this equality from Corollary 8.11. Indeed, Proposition 10.1 states that if  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  is the Dolbeault–Dirac operator on  $M$ , coupled to  $L^\omega$ , then the operator  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$  from Corollary 8.11 is precisely the Dolbeault–Dirac operator on the quotient  $M/\Gamma$  coupled to the line bundle  $L^\omega/\Gamma$ . In Section 10.3 we prove this proposition, and hence (10.8).

## 10.2 The isomorphism

The main step in our proof of (10.8) is the following proposition. We hope that the use of the letter  $\Gamma$  to denote a both discrete group and a space of sections will not cause any confusion.

**Proposition 10.1.** *Consider the Dolbeault–Dirac operator  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  on  $\Omega^{0,*}(M; L^\omega)$ , and the induced operator  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$  on*

$$\Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma),$$

*as defined in (8.9). There is an isomorphism*

$$\Xi : \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) \rightarrow \Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma)$$

*that is isometric with respect to the  $L^2$ -inner product and intertwines the Dolbeault–Dirac operator  $\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*$  on  $\Omega^{0,*}(M/\Gamma; L^\omega/\Gamma)$  and the operator  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$ .*

Consequently,  $\Xi$  induces a unitary isomorphism between the corresponding  $L^2$ -spaces, which by Lemma 4.31 intertwines the bounded operators obtained from  $\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*$  and  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$  using a normalising function with compactly supported Fourier transform. Hence (10.8) follows, as

$$\begin{aligned} V_\Gamma([\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*]) &= [(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma] && \text{by Corollary 8.11} \\ &= [\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*] && \text{by Proposition 10.1.} \end{aligned}$$

The isomorphism of  $C^\infty(M/\Gamma)$ -modules  $\Xi$  in Proposition 10.1 is defined as follows. The quotient map  $p : M \rightarrow M/\Gamma$  induces the vector bundle homomorphism  $Tp : TM \rightarrow T(M/\Gamma)$ . Since  $Tp$  is invariant with respect to the action of  $\Gamma$  on  $TM$ , it descends to a vector bundle homomorphism

$$(Tp)^\Gamma : (TM)/\Gamma \rightarrow T(M/\Gamma).$$

Because the group  $\Gamma$  is discrete, this map is an isomorphism. This is the most important reason why we assume  $\Gamma$  to be discrete. We denote the transpose of the isomorphism  $(Tp)^\Gamma$  by

$$(T^*p)^\Gamma : T^*(M/\Gamma) \rightarrow (T^*M)/\Gamma.$$

This gives

$$\wedge(T^*p)^\Gamma : \wedge T^*(M/\Gamma) \rightarrow \wedge(T^*M)/\Gamma, \quad (10.9)$$

and since  $Tp$  intertwines the almost complex structures on  $TM$  and  $T(M/\Gamma)$ , we obtain

$$\wedge^{0,*}(T^*p)^\Gamma : \wedge^{0,*}T^*(M/\Gamma) \rightarrow \wedge^{0,*}(T^*M)/\Gamma. \quad (10.10)$$

On the spaces of smooth sections of the vector bundles in question, the isomorphisms (10.9) and (10.10) induce isomorphisms of  $C^\infty(M/\Gamma)$ -modules

$$\Psi : \Omega^*(M/\Gamma) \rightarrow \Gamma^\infty(M/\Gamma, (\wedge T^*M)/\Gamma); \quad (10.11)$$

$$\Psi^{0,*} : \Omega^{0,*}(M/\Gamma) \rightarrow \Gamma^\infty(M/\Gamma, (\wedge^{0,*}T^*M)/\Gamma). \quad (10.12)$$

Now the isomorphism  $\Xi$  is defined as

$$\begin{aligned} \Xi : \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) &\cong \\ \Omega^{0,*}(M/\Gamma) \otimes_{C^\infty(M/\Gamma)} \Gamma^\infty(M/\Gamma, L^\omega/\Gamma) &\xrightarrow{\Psi^{0,*} \otimes 1_{\Gamma^\infty(M/\Gamma, L^\omega/\Gamma)}} \\ \Gamma^\infty(M/\Gamma, (\wedge^{0,*}T^*M)/\Gamma) \otimes_{C^\infty(M/\Gamma)} \Gamma^\infty(M/\Gamma, L^\omega/\Gamma) & \\ \cong \Gamma^\infty(M/\Gamma, (\wedge^{0,*}T^*M \otimes L^\omega)/\Gamma). & \end{aligned}$$

It is isometric by definition of the measure  $d\mathcal{O}$  on  $M/\Gamma$ , defined in (8.1), and the metrics on the vector bundles involved. An equivalent definition of the measure  $d\mathcal{O}$  is

$$\int_{M/\Gamma} f(\mathcal{O}) d\mathcal{O} := \int_U f(m) dm,$$

for  $f \in C(M/\Gamma)$ , where  $U \subset M$  is any fundamental domain of the  $\Gamma$ -action. Here by a fundamental domain, we mean an open subset  $U \subset M$  such that  $\Gamma \cdot U$  is dense in  $M$ , and that for all  $\gamma \in \Gamma$  and  $m \in U$ ,

$$\gamma \cdot m \in U \quad \Rightarrow \quad \gamma = e.$$

It remains to prove that  $\Xi$  intertwines the operators  $\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*$  and  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$ .

## 10.3 Proof of Proposition 10.1

### The connections

Let  $\psi_{L^\omega} : \Gamma^\infty(M, L^\omega)^\Gamma \rightarrow \Gamma^\infty(M/\Gamma, L^\omega/\Gamma)$  be the isomorphism of  $C^\infty(M)^\Gamma \cong C^\infty(M/\Gamma)$ -modules from Proposition 8.6, with  $E = L^\omega$  and  $H = \Gamma$ . Also consider the pullback  $p^*$  of differential forms on  $M/\Gamma$  to invariant differential forms on  $M$ . It defines an isomorphism of  $C^\infty(M/\Gamma) \cong C^\infty(M)^\Gamma$ -modules

$$p^* : \Omega^*(M/\Gamma) \rightarrow \Omega^*(M)^\Gamma.$$

The prequantum connection  $\nabla^\Gamma$  on the prequantum line bundle  $L^\omega/\Gamma \rightarrow M/\Gamma$  is defined by the property that  $p^*\nabla^\Gamma = \nabla$  (see Section 3.6). Explicitly, this definition can be expressed by

commutativity of the following diagram:

$$\begin{array}{ccc}
 \Omega^*(M; L^\omega)^\Gamma & \xrightarrow{\nabla} & \Omega^*(M; L^\omega)^\Gamma \\
 \downarrow \cong & & \downarrow \cong \\
 \Omega^*(M)^\Gamma \otimes_{C^\infty(M)^\Gamma} \Gamma^\infty(M, L^\omega)^\Gamma & & \Omega^*(M)^\Gamma \otimes_{C^\infty(M)^\Gamma} \Gamma^\infty(M, L^\omega)^\Gamma \\
 \uparrow \cong & & \uparrow \cong \\
 \Omega^*(M/\Gamma) \otimes_{C^\infty(M/\Gamma)} \Gamma^\infty(M/\Gamma, L^\omega/\Gamma) & & \Omega^*(M/\Gamma) \otimes_{C^\infty(M/\Gamma)} \Gamma^\infty(M/\Gamma, L^\omega/\Gamma) \\
 \uparrow \cong & & \uparrow \cong \\
 \Omega^*(M/\Gamma; L^\omega/\Gamma) & \xrightarrow{\nabla^\Gamma} & \Omega^*(M/\Gamma; L^\omega/\Gamma).
 \end{array} \tag{10.13}$$

By definition of the almost complex structure on  $T(M/\Gamma)$ , we have

$$p^*(\Omega^{0,q}(M/\Gamma)) = \Omega^{0,q}(M)^\Gamma$$

for all  $q$ . Therefore, commutativity of diagram (10.13) implies that the following diagram commutes:

$$\begin{array}{ccc}
 \Omega^{0,*}(M; L^\omega)^\Gamma & \xrightarrow{\bar{\partial}_{L^\omega}} & \Omega^{0,*}(M; L^\omega)^\Gamma \\
 \uparrow \cong & & \uparrow \cong \\
 \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) & \xrightarrow{\bar{\partial}_{L^\omega/\Gamma}} & \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma),
 \end{array} \tag{10.14}$$

with  $\bar{\partial}_{L^\omega}$  and  $\bar{\partial}_{L^\omega/\Gamma}$  as in Definition 3.19.

## The Dirac operators

By definition of the measure  $d\mathcal{O}$  on  $M/\Gamma$ , the metric  $g^\Gamma$  on  $T(M/\Gamma)$  induced by the metric  $g = \omega(-, J-)$  on  $TM$ , and the metric  $(-, -)_{L^\omega/\Gamma}$  on  $L^\omega/\Gamma$ , induced by the metric  $(-, -)_{L^\omega}$  on  $L^\omega$ , the isomorphism

$$p^* \otimes \psi_{L^\omega}^{-1} : \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) \rightarrow \Omega^{0,*}(M; L^\omega)^\Gamma$$

is isometric with respect to the inner product on  $\Omega^{0,*}(M/\Gamma; L^\omega/\Gamma)$  defined by

$$(\alpha \otimes \sigma, \beta \otimes \tau) = \int_{M/\Gamma} g^\Gamma(\alpha, \beta)(\sigma, \tau)_{L^\omega/\Gamma} d\mathcal{O}, \tag{10.15}$$

for all  $\alpha, \beta \in \Omega^{0,*}(M/\Gamma)$  and  $\sigma, \tau \in \Gamma^\infty(M/\Gamma, L^\omega/\Gamma)$ , and the inner product on  $\Omega^{0,*}(M; L^\omega)^\Gamma$  defined by

$$(\zeta \otimes s, \xi \otimes t) = \int_U g(\zeta, \xi)(m)(s, t)_{L^\omega}(m) dm, \tag{10.16}$$

for all  $\zeta, \xi \in \Omega^{0,*}(M)^\Gamma$  and  $s, t \in \Gamma^\infty(M, L^\omega)^\Gamma$ . (Recall that  $U \subset M$  is a fundamental domain for the  $\Gamma$ -action.)

In the definition of the Dolbeault–Dirac operator  $\bar{\partial}_{L^\omega/\Gamma} + (\bar{\partial}_{L^\omega/\Gamma})^*$  on  $M/\Gamma$ , the formal adjoint  $(\bar{\partial}_{L^\omega/\Gamma})^*$  is defined with respect to the inner product (10.15). If we denote the metric  $(-, -)_{L^\omega}$  on  $L^\omega$  by  $H^{L^\omega}$  for the moment, then the formal adjoint  $\bar{\partial}_{L^\omega}^*$  is defined by

$$\int_M (g \otimes H^{L^\omega}) (\bar{\partial}_{L^\omega}^* \eta, \theta) (m) dm = \int_M (g \otimes H^{L^\omega}) (\eta, \bar{\partial}_{L^\omega} \theta) (m) dm,$$

for all  $\eta, \theta \in \Omega^{0,*}(M; L^\omega)$ ,  $\theta$  with compact support. But this is actually the same as the formal adjoint of  $\bar{\partial}_{L^\omega}$  with respect to the inner product (10.16):

**Lemma 10.2.** *Let  $\Gamma$  be a discrete group, acting properly and freely on a manifold  $M$ , equipped with a  $\Gamma$ -invariant measure  $dm$ . Suppose  $M/\Gamma$  is compact. Let  $E \rightarrow M$  be a  $\Gamma$ -vector bundle, equipped with a  $\Gamma$ -invariant metric  $(-, -)_E$ . Let*

$$D : \Gamma^\infty(M, E) \rightarrow \Gamma^\infty(M, E)$$

*be a  $\Gamma$ -equivariant differential operator. Let*

$$D^* : \Gamma^\infty(M, E) \rightarrow \Gamma^\infty(M, E)$$

*be the operator such that for all  $s, t \in \Gamma^\infty(M, E)$ ,  $t$  with compact support,*

$$\int_M (D^* s, t)_E(m) dm = \int_M (s, Dt)_E(m) dm.$$

*Let  $U \subset M$  be a fundamental domain for the  $\Gamma$ -action. Then the restriction of  $D^*$  to  $\Gamma^\infty(M, E)^\Gamma$  satisfies*

$$\int_U (D^* s, t)_E(m) dm = \int_U (s, Dt)_E(m) dm, \quad (10.17)$$

*for all  $s, t \in \Gamma^\infty(M, E)^\Gamma$ .*

*Proof.* We will show that for all  $s \in \Gamma^\infty(M, E)^\Gamma$ , and all  $t$  in a dense subspace of  $\Gamma^\infty(M, E)^\Gamma$ , the equality (10.17) holds. Let  $\tau$  be a section of  $E$ , with compact support in  $U$ . Define the section  $t$  of  $E$  by extending the restriction  $\tau|_U$   $\Gamma$ -invariantly to  $M$ . The space of all sections  $t$  obtained in this way is dense in  $\Gamma^\infty(M, E)^\Gamma$  with respect to the topology induced by the inner product

$$(s, t) := \int_U (s, t)_E(m) dm$$

used in (10.17).

Then for all  $s \in \Gamma^\infty(M, E)^\Gamma$ ,

$$\begin{aligned} \int_U (D^* s, t)_E(m) dm &= \int_M (D^* s, \tau)_E(m) dm \\ &= \int_M (s, D\tau)_E(m) dm \\ &= \int_U (s, Dt)_E(m) dm. \end{aligned}$$

□

We conclude that  $p^* \otimes \psi_{L^\omega}^{-1}$  is an isometric isomorphism with respect to the inner products used to define the adjoints  $\bar{\partial}_{L^\omega}^*$  and  $(\bar{\partial}_{L^\omega/\Gamma})^*$ . Hence the commutativity of diagram (10.14) implies:

**Corollary 10.3.** *The following diagram commutes:*

$$\begin{array}{ccc} \Omega^{0,*}(M; L^\omega)^\Gamma & \xrightarrow{\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*} & \Omega^{0,*}(M; L^\omega)^\Gamma \\ p^* \otimes \psi_{L^\omega}^{-1} \uparrow \cong & & p^* \otimes \psi_{L^\omega}^{-1} \uparrow \cong \\ \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) & \xrightarrow{\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*} & \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma). \end{array}$$

**Remark 10.4.** Corollary 10.3 shows that for free actions by discrete groups, a much stronger statement than the Guillemin–Sternberg–Landsman conjecture holds. Indeed, by Remark 8.12 the Guillemin–Sternberg conjecture states that the restriction of the operator  $\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*$  to  $\Omega^{0,*}(M; L^\omega)^\Gamma$  is related to the operator  $\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*$  by the fact that their indices are equal (as operators on smooth, not necessarily  $L^{\omega^2}$ , sections). But these operators are in fact more strongly related: they are intertwined by an isometric isomorphism.

## End of the proof of Proposition 10.1

The last step in the proof of Proposition 10.1 is a decomposition of the isomorphism

$$p^* : \Omega^*(M/\Gamma) \rightarrow \Omega^*(M)^\Gamma.$$

**Lemma 10.5.** *The following diagram commutes:*

$$\begin{array}{ccc} \Omega^*(M/\Gamma) & \xrightarrow[p^*]{\cong} & \Omega^*(M)^\Gamma \\ \Psi \downarrow \cong & \swarrow \psi_{\wedge T^*M} & \\ \Gamma^\infty(M/\Gamma, (\wedge T^*M)/\Gamma), & & \end{array}$$

where  $\Psi$  is the isomorphism (10.11), and  $\psi_{\wedge T^*M}$  is the isomorphism from Proposition 8.6.

The proof of this lemma is a short and straightforward computation.

*Proof of Proposition 10.1.* Together with Lemma 10.5 and the definition of the operator

$$(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma : \Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma) \rightarrow \Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma),$$

Corollary 10.3 implies that the following diagram commutes:

$$\begin{array}{ccc} \Omega^{0,*}(M; L^\omega)^\Gamma & \xrightarrow{\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*} & \Omega^{0,*}(M; L^\omega)^\Gamma \\ \psi_{\wedge^{0,*} T^*M} \otimes \psi_{L^\omega} \downarrow \cong & (\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma & \cong \downarrow \psi_{\wedge^{0,*} T^*M} \otimes \psi_{L^\omega} \\ \Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma) & \xrightarrow{(\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*)} & \Gamma^\infty(M/\Gamma, (\wedge^{0,*} T^*M \otimes L^\omega)/\Gamma) \\ \Xi = \Psi^{0,*} \otimes 1 \uparrow \cong & & \Xi = \Psi^{0,*} \otimes 1 \uparrow \cong \\ \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma) & \xrightarrow{\bar{\partial}_{L^\omega/\Gamma} + \bar{\partial}_{L^\omega/\Gamma}^*} & \Omega^{0,*}(M/\Gamma; L^\omega/\Gamma). \end{array}$$

Indeed, the outside diagram commutes by Corollary 10.3 and Lemma 10.5, and the upper square commutes by definition of  $(\bar{\partial}_{L^\omega} + \bar{\partial}_{L^\omega}^*)^\Gamma$ . Hence the lower square commutes as well, which is Proposition 10.1.  $\square$

# Chapter 11

## Special case: abelian discrete groups

We now consider the situation of Theorem 6.5, with the additional assumption that  $G = \Gamma$  is an abelian discrete group. Then the Guillemin–Sternberg conjecture can be proved directly, without using naturality of the assembly map (Theorem 7.1). This proof is based on Proposition 10.1, and the description of the assembly map in this special case given by Baum, Connes and Higson [8], Example 3.11 (which in turn is based on Lusztig [55]). We will first explain this example in a little more detail than given in [8], and then show how it implies Theorem 6.5 for abelian discrete groups.

This chapter only serves to illustrate Theorem 6.5, and the rest of this thesis does not depend on it. We have therefore chosen to give less detailed arguments in this chapter than in the other ones.

### 11.1 The assembly map for abelian discrete groups

The proof of the Guillemin–Sternberg conjecture for discrete abelian groups is based on the following result:

**Proposition 11.1.** *Let  $M$ ,  $E$ ,  $D$  and  $D^\Gamma$  be as in Section 8.4. Suppose that  $G = \Gamma$  is abelian and discrete. Using the normalising function  $b(x) = \frac{x}{\sqrt{1+x^2}}$ , we form the operator  $F := b(D)$ , so that we have the class*

$$[L^2(M, E), F, \pi^M] \in K_0^\Gamma(M).$$

Then<sup>1</sup>

$$R_\Gamma^0 \circ \mu_M^\Gamma [L^2(M, E), F] = \text{index } D^\Gamma.$$

In view of Proposition 10.1, Proposition 11.1 implies our Guillemin–Sternberg conjecture (i.e. Theorem 6.5) for discrete abelian groups.

### Kernels of operators as vector bundles

Using Example 3.11 from [8], we can explicitly compute

$$[\mathcal{E}, F_\mathcal{E}] := \mu_M^\Gamma [L^2(M, E), F] \in KK_0(\mathbb{C}, C^*(\Gamma)). \quad (11.1)$$

---

<sup>1</sup>Recall that we use  $\text{index } D^\Gamma$  to denote the formal difference of the even and odd parts of the kernel of  $D^\Gamma$ .

Note that since the group  $\Gamma$  is discrete, its unitary dual  $\hat{\Gamma}$  is compact. And because  $\Gamma$  is abelian, all irreducible unitary representations are of the form

$$U_\alpha : \Gamma \rightarrow \mathbf{U}(1),$$

for  $\alpha \in \hat{\Gamma}$ . Fourier transform defines an isomorphism  $C^*(\Gamma) \cong C_0(\hat{\Gamma})$ . Therefore,

$$KK_0(\mathbb{C}, C^*(\Gamma)) \cong K_0(C^*(\Gamma)) \cong K_0(C_0(\hat{\Gamma})) \cong K^0(\hat{\Gamma}).$$

Because  $\hat{\Gamma}$  is compact, the image of  $[\mathcal{E}, F_{\mathcal{E}}]$  in  $K^0(\hat{\Gamma})$  is the difference of the isomorphism classes of two vector bundles over  $\hat{\Gamma}$ . These two vector bundles can be determined as follows. For all  $\alpha \in \hat{\Gamma}$ , we define the Hilbert space  $\mathcal{H}_\alpha$  as the space of all measurable sections  $s_\alpha$  of  $E$  (modulo equality almost everywhere), such that for all  $\gamma \in \Gamma$ ,

$$\gamma \cdot s_\alpha = U_\alpha(\gamma)^{-1} s_\alpha,$$

and such that the norm

$$\|s_\alpha\|_\alpha^2 = (s_\alpha, s_\alpha)_\alpha \quad (11.2)$$

is finite, where the inner product  $(-, -)_\alpha$  is defined by

$$(s_\alpha, t_\alpha)_\alpha := \int_{M/\Gamma} (s_\alpha(\varphi(\mathcal{O})), t_\alpha(\varphi(\mathcal{O})))_E d\mathcal{O},$$

where  $\varphi$  is any measurable section of the principal fibre bundle  $M \rightarrow M/\Gamma$ . The space  $\mathcal{H}_\alpha$  is isomorphic to the space of  $L^2$ -sections of the vector bundle  $E_\alpha$ , where

$$E_\alpha := E / (\gamma \cdot e \sim U_\alpha^{-1}(\gamma)e) \rightarrow M/\Gamma.$$

Let  $\mathcal{H}_\alpha^D$  be the dense subspace

$$\mathcal{H}_\alpha^D := \{s_\alpha \in \mathcal{H}_\alpha \cap \Gamma^\infty(M, E); Ds_\alpha \in \mathcal{H}_\alpha\} \subset \mathcal{H}_\alpha. \quad (11.3)$$

Because the operator  $D$  is  $\Gamma$ -equivariant, it restricts to an unbounded operator

$$D_\alpha : \mathcal{H}_\alpha^D \rightarrow \mathcal{H}_\alpha$$

on  $\mathcal{H}_\alpha$ . It is essentially self-adjoint by [34], Corollary 10.2.6., and hence induces the bounded operator

$$F_\alpha := \frac{D_\alpha}{\sqrt{1 + D_\alpha^2}} \in \mathcal{B}(\mathcal{H}_\alpha). \quad (11.4)$$

The grading on  $E$  induces a grading on  $\mathcal{H}_\alpha$  with respect to which  $D_\alpha$  and  $F_\alpha$  are odd. The operators  $F_\alpha$  are elliptic pseudo-differential operators:

**Lemma 11.2.** *Let  $D$  be an elliptic, first order differential operator on a vector bundle  $E \rightarrow M$ , and suppose  $D$  defines an essentially self-adjoint operator on  $L^2(M, E)$  with respect to some measure on  $M$  and some metric on  $E$ . Then the operator  $F := \frac{D}{\sqrt{1+D^2}}$  is an elliptic pseudo-differential operator.*



*Proof.* It is sufficient to show that  $(1 + D^2)^{-\frac{1}{2}}$  is a pseudo-differential operator. According to [9], a bounded operator  $A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  is a pseudo-differential operator on  $\mathbb{R}^n$  if and only if all iterated commutators with  $x_j$  (as a multiplication operator) and  $\frac{\partial}{\partial x_j}$  are bounded operators. This immediately yields the lemma for  $M = \mathbb{R}^n$  (cf. [9], Theorem 4.2). To extend this result to the manifold case, we recall that an operator  $A : C^\infty(M) \rightarrow \mathcal{D}'(M)$  on a manifold  $M$  is a pseudo-differential operator when for each choice of smooth functions  $f, g$  with support in a single coordinate neighbourhood,  $fAg$  is a pseudo-differential operator on  $\mathbb{R}^n$ . (Here one has to admit nonconnected coordinate neighbourhoods.)

Now write  $(1 + D^2)^{-\frac{1}{2}}$  as a Dunford integral (cf. [21], pp. 556–577), as follows:

$$(1 + D^2)^{-\frac{1}{2}} = \frac{1}{2\pi i} \oint_C (1 + z)^{-\frac{1}{2}} (z - D^2)^{-1} dz.$$

Here  $C$  is any contour around the spectrum of  $D$ . To compute the commutators of  $f(1 + D^2)^{-\frac{1}{2}}g$  with  $x_j$  and  $\frac{\partial}{\partial x_j}$ , one may take these inside the contour integral. Boundedness of all iterated commutators then easily follows, using the fact that  $f$  and  $g$  have compact support.

The same argument, with the exponent  $-\frac{1}{2}$  replaced by  $\frac{1}{2}$ , shows that  $(1 + D^2)^{\frac{1}{2}}$  is a pseudo-differential operator, and ellipticity of  $(1 + D^2)^{-\frac{1}{2}}$  follows.  $\square$

We were informed of the above proof by Elmar Schrohe. An independent proof of this lemma was suggested to us by John Roe, who mentioned that in the case at hand the functional calculus for (pseudo-)differential operators developed in [78] for compact manifolds may be extended to the noncompact case. A third proof may be constructed using heat kernel techniques, as in the unpublished Diplomarbeit of Hanno Sahlmann (Rainer Verch, private communication).

Consider the field of Hilbert spaces

$$(\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}} \rightarrow \hat{\Gamma}. \quad (11.5)$$

In the next section, we will give this field the structure of a *continuous* field of Hilbert spaces by specifying its space of continuous sections  $\Gamma(\hat{\Gamma}, (\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}})$ . Consider the subfields

$$\begin{aligned} (\ker D_\alpha^+)_{\alpha \in \hat{\Gamma}} &\rightarrow \hat{\Gamma}; \\ (\ker D_\alpha^-)_{\alpha \in \hat{\Gamma}} &\rightarrow \hat{\Gamma}. \end{aligned} \quad (11.6)$$

These are indeed well-defined subfields of  $(\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}}$  because  $\ker D_\alpha^\pm = \ker F_\alpha^\pm$  by the elliptic regularity theorem (here we use Lemma 11.2), and by the fact that the operator  $\frac{1}{\sqrt{1 + D_\alpha^- D_\alpha^+}}$  is invertible.

Suppose that the fields (11.6) are vector bundles over  $\hat{\Gamma}$  in the topology on (11.5) that we will define in Section 11.2. As in the proof that  $KK_0(\mathbb{C}, C(\hat{\Gamma})) \cong K_0(C(\hat{\Gamma}))$  (see the remark below Theorem 5.12), the operator  $D$  can always be replaced by an operator for which  $\Gamma((\ker D_\alpha^\pm)_{\alpha \in \hat{\Gamma}})$  are finitely generated projective  $C(\hat{\Gamma})$ -modules, that is, for which  $(\ker D_\alpha^\pm)_{\alpha \in \hat{\Gamma}}$  are vector bundles, and that the  $K$ -theory class

$$\mu_M^\Gamma [L^2(M, E), F] \in KK(\mathbb{C}, C(\hat{\Gamma}))$$

is the same, whether we make this replacement or not.

Then:

**Proposition 11.3.** *The image of the class  $[L^2(M, E), F] \in K_0^\Gamma(M)$  under the assembly map  $\mu_M^\Gamma$  is*

$$\mu_M^\Gamma [L^2(M, E), F] = \left[ (\ker D_\alpha^+)_{\alpha \in \hat{\Gamma}} \right] - \left[ (\ker D_\alpha^-)_{\alpha \in \hat{\Gamma}} \right] \in K^0(\hat{\Gamma}).$$

Proposition 11.3 will be proved in the next two sections.

## 11.2 The Hilbert $C^*$ -module part of the assembly map

In this section we determine the Hilbert  $C^*(\Gamma) \cong C_0(\hat{\Gamma})$ -module  $\mathcal{E}$  in (11.1). The result is Proposition 11.7.

### A unitary isomorphism

Let  $d\alpha$  be the measure on  $\hat{\Gamma}$  corresponding to the counting measure on  $\Gamma$  via the Fourier transform. Consider the Hilbert space

$$\mathcal{H} := \int_{\hat{\Gamma}}^{\oplus} \mathcal{H}_\alpha d\alpha.$$

That is,  $\mathcal{H}$  consists of the measurable maps

$$\begin{aligned} s : \hat{\Gamma} &\rightarrow (\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}}; \\ \alpha &\mapsto s_\alpha, \end{aligned}$$

such that  $s_\alpha \in \mathcal{H}_\alpha$  for all  $\alpha$ , and

$$\|s\|_{\mathcal{H}}^2 = (s, s)_{\mathcal{H}} := \int_{\hat{\Gamma}} \|s_\alpha\|_\alpha^2 d\alpha < \infty.$$

Define the linear map  $V : \mathcal{H} \rightarrow L^2(M, E)$  by

$$(Vs)(m) := \int_{\hat{\Gamma}} s_\alpha(m) d\alpha.$$

**Lemma 11.4.** *The map  $V$  is a unitary isomorphism, with inverse*

$$(V^{-1}\sigma)_\alpha(m) = \sum_{\gamma \in \Gamma} \gamma \cdot \sigma(\gamma^{-1}m) U_\alpha(\gamma), \quad (11.7)$$

for all  $\sigma \in \Gamma_c(M, E) \subset L^2(M, E)$ .

**Remark 11.5.** It follows from unitarity of  $V$  that  $Vs$  is indeed an  $L^2$ -section of  $E$  for all  $s \in \mathcal{H}$ . Conversely, a direct computation shows that for all  $\sigma \in L^2(M, E)$ ,  $\alpha \in \hat{\Gamma}$  and  $\gamma \in \Gamma$ , one has

$$\gamma \cdot (V^{-1}\sigma)_\alpha = U_\alpha(\gamma)^{-1} (V^{-1}\sigma)_\alpha,$$

so that  $V^{-1}\sigma$  lies in  $\mathcal{H}$ .

*Sketch of proof of Lemma 11.4.* The proof is based on the observations that for all  $\alpha \in \hat{\Gamma}$ ,

$$\sum_{\gamma \in \Gamma} U_\alpha(\gamma) = \delta_1(\alpha), \quad (11.8)$$

where  $\delta_1 \in \mathcal{D}'(\hat{\Gamma})$  is the  $\delta$ -distribution at the trivial representation  $1 \in \hat{\Gamma}$ , and that for all  $\gamma \in \Gamma$ ,

$$\int_{\hat{\Gamma}} U_\alpha(\gamma) d\alpha = \delta_{\gamma e}, \quad (11.9)$$

the Kronecker delta of  $\gamma$  and the identity element. Using these facts, one can easily verify that  $V$  is an isometry, and that (11.7) is indeed the inverse of  $V$ .  $\square$

The representation  $\pi_{\mathcal{H}}$  of  $\Gamma$  in  $\mathcal{H}$  corresponding to the standard representation (3.4) of  $\Gamma$  in  $L^2(M, E)$  via the isomorphism  $V$  is given by

$$(\pi_{\mathcal{H}}(\gamma)s)_\alpha = U_\alpha(\gamma)^{-1} s_\alpha.$$

This follows directly from the definitions of the space  $\mathcal{H}_\alpha$  and the map  $V$ .

## Fourier transform

By definition of the assembly map, the Hilbert  $C^*(\Gamma)$ -module  $\mathcal{E}$  is the closure of the space  $\Gamma_c(M, E)$  in the norm

$$\|\sigma\|_{\mathcal{E}}^2 := \|\gamma \mapsto (\sigma, \gamma \cdot \sigma)_{L^2(M, E)}\|_{C^*(\Gamma)}.$$

The  $C^*(\Gamma)$ -module structure of  $\mathcal{E}$  is defined by

$$f \cdot \sigma = \sum_{\gamma \in \Gamma} f(\gamma) \gamma \cdot \sigma,$$

for all  $f \in C_c(\Gamma)$  and  $\sigma \in \Gamma_c(M, E)$ . The isomorphism  $V$  induces an isomorphism of the Hilbert  $C^*(\Gamma)$ -module  $\mathcal{E}$  with the closure  $\mathcal{E}_{\mathcal{H}}$  of  $V^{-1}(\Gamma_c(M, E)) \subset \mathcal{H}$  in the norm

$$\|s\|_{\mathcal{E}_{\mathcal{H}}}^2 := \|\gamma \mapsto (Vs, \gamma \cdot Vs)_{L^2(M, E)}\|_{C^*(\Gamma)} = \|\gamma \mapsto (s, \pi_{\mathcal{H}}(\gamma)s)_{\mathcal{H}}\|_{C^*(\Gamma)},$$

by unitarity of  $V$ . The  $C^*(\Gamma)$ -module structure on  $\mathcal{E}_{\mathcal{H}}$  corresponding to the one on  $\mathcal{E}$  via  $V$  is given by

$$f \cdot s = \sum_{\gamma \in \Gamma} f(\gamma) \pi_{\mathcal{H}}(\gamma)s, \quad (11.10)$$

for all  $f \in C_c(\Gamma)$  and  $s \in V^{-1}(\Gamma_c(M, E))$ .

Next, we use the isomorphism  $C_0(\hat{\Gamma}) \cong C^*(\Gamma)$  defined by the Fourier transform  $\psi \mapsto \hat{\psi}$ , where

$$\hat{\psi}(\gamma) = \int_{\hat{\Gamma}} \psi(\alpha) U_\alpha(\gamma) d\alpha$$

for all  $\psi \in C_c(\hat{\Gamma})$ . Because of (11.8) and (11.9), the inverse Fourier transform is given by  $f \mapsto \hat{f}$ , where for  $f \in C_c(\Gamma)$ , one has

$$\hat{f}(\alpha) = \sum_{\gamma \in \Gamma} f(\gamma) U_\alpha(\gamma)^{-1}.$$

So via the Fourier transform, the Hilbert  $C^*(\Gamma)$ -module  $\mathcal{E}_{\mathcal{H}}$  corresponds to the Hilbert  $C_0(\hat{\Gamma})$ -module  $\hat{\mathcal{E}}_{\mathcal{H}}$ , which is the closure of the space  $V^{-1}(\Gamma_c(M, E))$  in the norm

$$\begin{aligned} \|s\|_{\hat{\mathcal{E}}_{\mathcal{H}}}^2 &= \left\| \alpha \mapsto \sum_{\gamma \in \Gamma} (s, \pi_{\mathcal{H}}(\gamma)s)_{\mathcal{H}} U_{\alpha}(\gamma)^{-1} \right\|_{C_0(\hat{\Gamma})}^2 \\ &= \sup_{\alpha \in \hat{\Gamma}} \left| \sum_{\gamma \in \Gamma} (s, \pi_{\mathcal{H}}(\gamma)s)_{\mathcal{H}} U_{\alpha}(\gamma)^{-1} \right|. \end{aligned} \quad (11.11)$$

### Continuous sections

Using the following lemma, we will describe the Hilbert  $C_0(\hat{\Gamma})$ -module  $\hat{\mathcal{E}}_{\mathcal{H}}$  as the space of continuous sections of a continuous field of Hilbert spaces.

**Lemma 11.6.** *For all  $s, t \in V^{-1}(\Gamma_c(M, E))$ ,*

$$\sum_{\gamma \in \Gamma} (s, \pi_{\mathcal{H}}(\gamma)t)_{\mathcal{H}} U_{\alpha}(\gamma)^{-1} = (s_{\alpha}, t_{\alpha})_{\alpha}.$$

*Proof.* Let  $\varphi$  be a measurable section of the principal fibre bundle  $M \rightarrow M/\Gamma$ . Then by (11.8),

$$\begin{aligned} \sum_{\gamma \in \Gamma} (s, \pi_{\mathcal{H}}(\gamma)t)_{\mathcal{H}} U_{\alpha}(\gamma)^{-1} &= \\ \sum_{\gamma \in \Gamma} \left( \int_{\hat{\Gamma}} \int_{M/\Gamma} (s_{\beta}(\varphi(\mathcal{O})), U_{\beta}(\gamma)^{-1} t_{\beta}(\varphi(\mathcal{O})))_E d\mathcal{O} d\beta \right) U_{\alpha}(\gamma)^{-1} &= \\ \int_{M/\Gamma} (s_{\alpha}(\varphi(\mathcal{O})), t_{\alpha}(\varphi(\mathcal{O})))_E d\mathcal{O} &= (s_{\alpha}, t_{\alpha})_{\alpha}. \end{aligned}$$

□

We conclude from (11.11) and Lemma 11.6 that  $\hat{\mathcal{E}}_{\mathcal{H}}$  is the closure of  $V^{-1}(\Gamma_c(M, E))$  in the norm

$$\|s\|_{\hat{\mathcal{E}}_{\mathcal{H}}}^2 = \sup_{\alpha \in \hat{\Gamma}} \|s_{\alpha}\|_{\alpha}^2.$$

Therefore, it makes sense to *define* the space  $\Gamma(\hat{\Gamma}, (\mathcal{H}_{\alpha})_{\alpha \in \hat{\Gamma}})$  of continuous sections of the field of Hilbert spaces  $(\mathcal{H}_{\alpha})_{\alpha \in \hat{\Gamma}}$  as the  $C_0(\hat{\Gamma})$ -module  $\hat{\mathcal{E}}_{\mathcal{H}}$  (cf. [19, 77]). Then our construction implies

**Proposition 11.7.** *The Hilbert  $C^*(\Gamma)$ -module  $\mathcal{E}$  is isomorphic to the Hilbert  $C_0(\hat{\Gamma})$ -module  $\Gamma(\hat{\Gamma}, (\mathcal{H}_{\alpha})_{\alpha \in \hat{\Gamma}})$ .*

Let us verify explicitly that the representations of  $C_0(\hat{\Gamma})$  in  $\hat{\mathcal{E}}_{\mathcal{H}}$  and in  $\Gamma(\hat{\Gamma}, (\mathcal{H}_{\alpha})_{\alpha \in \hat{\Gamma}})$  are indeed intertwined by the isomorphism induced by  $V$  and the Fourier transform: for all  $f \in C_c(\Gamma)$  and all  $s \in V^{-1}(\Gamma_c(M, E))$ , we have

$$\begin{aligned} (f \cdot s)_{\alpha} &= \sum_{\gamma \in \Gamma} f(\gamma) (\pi_{\mathcal{H}}(\gamma)s)_{\alpha} \quad \text{by (11.10)} \\ &= \sum_{\gamma \in \Gamma} f(\gamma) U_{\alpha}(\gamma)^{-1} s_{\alpha} \\ &= \hat{f}(\alpha) s_{\alpha}. \end{aligned}$$

## 11.3 The operator part of the assembly map

**Proposition 11.8.** *Consider the adjointable operator  $F_{\hat{\mathcal{E}}\mathcal{H}} = (F_\alpha)_{\alpha \in \hat{\Gamma}}$  on the Hilbert  $C_0(\hat{\Gamma})$ -module  $\hat{\mathcal{E}}\mathcal{H} = \Gamma(\hat{\Gamma}, (\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}})$ , given by*

$$(F_{\hat{\mathcal{E}}\mathcal{H}} s)_\alpha := F_\alpha s_\alpha,$$

*for all  $\alpha \in \hat{\Gamma}$  and  $s \in \Gamma(\hat{\Gamma}, (\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}})$ . Here  $F_\alpha$  is the operator (11.4). Then for all  $s \in V^{-1}(\Gamma_c(M, E))$ , we have*

$$FVs = VF_{\hat{\mathcal{E}}\mathcal{H}} s.$$

*Proof.* The claim is that for all such  $s$ , and all  $m \in M$ ,

$$FVs(m) = \int_{\hat{\Gamma}} F_\alpha s_\alpha(m) d\alpha.$$

Let  $\mathcal{H}^D \subset \mathcal{H}$  be the space of  $s \in \mathcal{H}$  such that  $Vs \in \Gamma_c^\infty(M, E)$ , and  $s_\alpha \in \mathcal{H}_\alpha^D$  for all  $\alpha \in \hat{\Gamma}$  (see (11.3)).

Note that we have  $DVs(m) = \int_{\hat{\Gamma}} Ds_\alpha(m) d\alpha$  for all  $s \in \mathcal{H}^D$  and  $m \in M$ . Because of Lemma 4.31 this proves the proposition, since  $\mathcal{H}^D$  is dense in  $\mathcal{H}$ .  $\square$

*Proof of Proposition 11.3.* Since  $\Gamma_c(M, E)$  is dense in  $\mathcal{E}$  and  $V^{-1}(\Gamma_c(M, E))$  is dense in  $\hat{\mathcal{E}}\mathcal{H}$ , Propositions 11.7 and 11.8 imply that

$$\begin{aligned} \mu_M^\Gamma [L^2(M, E), F] &= [\mathcal{E}, F_\mathcal{E}] \\ &= [\hat{\mathcal{E}}\mathcal{H}, F_{\hat{\mathcal{E}}\mathcal{H}}] \\ &= [\Gamma(\hat{\Gamma}, (\mathcal{H}_\alpha)_{\alpha \in \hat{\Gamma}}), (F_\alpha)_{\alpha \in \hat{\Gamma}}] \in KK_0(\mathbb{C}, C_0(\hat{\Gamma})). \end{aligned}$$

The image of this class in  $K_0(C_0(\hat{\Gamma}))$  is the formal difference of projective  $C_0(\hat{\Gamma})$ -modules

$$[\ker((F_\alpha^+)_{\alpha \in \hat{\Gamma}})] - [\ker((F_\alpha^-)_{\alpha \in \hat{\Gamma}})]. \quad (11.12)$$

By compactness of  $M/\Gamma$  and the elliptic regularity theorem, the kernels of  $F_\alpha^+$  and  $F_\alpha^-$  are equal to the kernels of  $D_\alpha^+$  and  $D_\alpha^-$ , respectively. By the remark above Proposition 11.3, we may suppose that the kernels of  $D_\alpha^+$  and  $D_\alpha^-$  define vector bundles over  $\hat{\Gamma}$ . Then by Lemma 11.9 below, the class (11.12) equals

$$[\Gamma(\hat{\Gamma}, (\ker D_\alpha^+)_{\alpha \in \hat{\Gamma}})] - [\Gamma(\hat{\Gamma}, (\ker D_\alpha^-)_{\alpha \in \hat{\Gamma}})].$$

Under the isomorphism  $K_0(C_0(\hat{\Gamma})) \cong K^0(\hat{\Gamma})$ , the latter class corresponds to

$$[(\ker D_\alpha^+)_{\alpha \in \hat{\Gamma}}] - [(\ker D_\alpha^-)_{\alpha \in \hat{\Gamma}}] \in K^0(\hat{\Gamma}).$$

$\square$

**Lemma 11.9.** *Let  $\mathcal{H}$  be a continuous field of Hilbert spaces over a topological space  $X$ , and let  $\Delta$  be its space of continuous sections. Let  $\mathcal{H}'$  be a subset of  $\mathcal{H}$  such that for all  $x \in X$ ,  $\mathcal{H}'_x := \mathcal{H}_x \cap \mathcal{H}'$  is a linear subspace of  $\mathcal{H}_x$ . Set*

$$\Delta' := \{s \in \Delta; s(x) \in \mathcal{H}'_x \text{ for all } x \in X\}.$$

*Let  $s : X \rightarrow \mathcal{H}'$  be a section. Then  $s$  is continuous in the subspace topology of  $\mathcal{H}'$  in  $\mathcal{H}$  if and only if  $s \in \Delta'$ .*

*Proof.* Let  $s : X \rightarrow \mathcal{H}$  be a section. Then  $s$  is a continuous section of  $\mathcal{H}'$  in the subspace topology if and only if  $s$  is a continuous section of  $\mathcal{H}$  and  $s(x) \in \mathcal{H}'_x$  for all  $x$ . The topology on  $\mathcal{H}$  is defined in such a way that  $s$  is continuous if and only if  $s \in \Delta$  [19, 77].  $\square$

## 11.4 Reduction

We will now describe the reduction map  $R_\Gamma^0 : K_0(C^*(\Gamma)) \rightarrow \mathbb{Z}$ , and prove Proposition 11.1.

**Lemma 11.10.** *Let  $\Gamma$  be an abelian discrete group, and let  $i : \{1\} \hookrightarrow \hat{\Gamma}$  be the inclusion of the trivial representation. The following diagram commutes:*

$$\begin{array}{ccc} K_0(C^*(\Gamma)) & \xrightarrow{R_\Gamma^0} & K_0(\mathbb{C}) \\ \cong \downarrow & & \downarrow \cong \\ K^0(\hat{\Gamma}) & \xrightarrow{i^*} & K^0(\{1\}). \end{array}$$

That is,

$$R_\Gamma^0([E]) = \dim E_1 = \text{rank}(E) \in \mathbb{Z},$$

for all vector bundles  $E \rightarrow \hat{\Gamma}$ .

The proof is a straightforward verification.

*End of proof of Proposition 11.1.* From Lemma 11.10 and Proposition 11.3, we conclude that

$$R_\Gamma^0 \circ \mu_M^\Gamma [L^2(M, E), F] = [\ker D_1^+] - [\ker D_1^-] = \text{index } D_1 \in \mathbb{Z}.$$

The Hilbert space  $\mathcal{H}_1$  is isomorphic to  $L^2(M/\Gamma, E/\Gamma)$ , and this isomorphism intertwines  $D_1$  and  $D^\Gamma$ . Hence Proposition 11.1 follows.  $\square$

## 11.5 Example: the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n}$

For some natural number  $n$ , let  $M$  be the manifold  $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n} \cong \mathbb{C}^n$ . An element of  $M$  is denoted by  $(q, p) := (q_1, p_1, \dots, q_n, p_n)$ , where  $q_j, p_j \in \mathbb{R}$ , or by  $q + ip = z := (z_1, \dots, z_n)$ , where  $z_j = q_j + ip_j \in \mathbb{C}$ . We equip  $M$  with the standard symplectic form  $\omega := \sum_{j=1}^n dp_j \wedge dq_j$ , as in (2.2).

Let  $\Gamma$  be the group  $\Gamma = \mathbb{Z}^{2n} \cong \mathbb{Z}^n + i\mathbb{Z}^n$ . The action of  $\Gamma$  on  $M$  by addition is denoted by  $\alpha$ . Our aim is to find a prequantisation for this action and the corresponding Dirac operator for general  $n$ , and the quantisation of this action for  $n = 1$ .

### Prequantisation

Let  $L := M \times \mathbb{C} \rightarrow M$  be the trivial line bundle. Inspired by the construction of line bundles on tori with a given Chern class (see e.g. [25], pp. 307–317), we lift the action of  $\Gamma$  on  $M$  to an action of  $\Gamma$  on  $L$  (still called  $\alpha$ ), by setting

$$\begin{aligned} e_j \cdot (z, w) &= (z + e_j, w); \\ ie_j \cdot (z, w) &= (z + ie_j, e^{-2\pi i z_j} w). \end{aligned}$$

Here  $z \in M$ ,  $w \in \mathbb{C}$ , and

$$e_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n,$$

the 1 being in the  $j$ th place. The corresponding representation of  $\Gamma$  in the space of smooth sections of  $L$  is denoted by  $\rho$ :

$$(\rho_{k+il}s)(z) = \alpha_{k+il}s(z - k - il),$$

for  $k, l \in \mathbb{Z}^n$  and  $z \in M$ . Define the metric  $(-, -)_L$  on  $L$  by

$$((z, w), (z, w'))_L = h(z)w\bar{w}',$$

where  $z \in M$ ,  $w, w' \in \mathbb{C}$ , and  $h \in C^\infty(M)$  is defined by

$$h(q + ip) := e^{2\pi i \sum_j (p_j - p_j^2)}.$$

Let  $\nabla$  be the connection on  $L$  defined by

$$\nabla := d + 2\pi i \sum_{j=1}^n p_j dz_j + \pi dp_j.$$

**Proposition 11.11.** *The triple  $(L, (-, -)_L, \nabla)$  defines an equivariant prequantisation for  $(M, \omega)$ .*

The proof of this proposition is a set of tedious computations. Because of the term  $2\pi i \sum_{j=1}^n p_j dq_j$  in the expression for the connection  $\nabla$ , it has the right curvature form. The terms  $-2\pi i \sum_{j=1}^n p_j dp_j$  and  $\pi dp_j$  do not change the curvature, and have been added to make  $\nabla$  equivariant. At the same time, the latter two terms ensure that there is a  $\Gamma$ -invariant metric (namely  $(-, -)_L$ ) with respect to which  $\nabla$  is Hermitian.

As we mentioned in Section 6.1, there is a procedure in [32] to lift the action of  $\mathbb{Z}^{2n}$  on  $\mathbb{R}^{2n}$  to a projective action on  $L$  that leaves the connection (for example)  $\nabla' := d + 2\pi i \sum_j p_j dq_j$  invariant. This projective action turns out to be an actual action in this case, and preserves the standard metric on  $L$ . We thus obtain prequantisation of this action that looks much simpler than the one given in this chapter. However, we found our formulas to be more suitable to compute the kernel of the associated Dirac operator.

## The Dirac operator

In this section, we compute the Dolbeault–Dirac operator  $\bar{\partial}_L + \bar{\partial}_L^*$  on  $M$ , coupled to  $L$ . We will simplify our notation by denoting this operator by  $\mathcal{D}$  in the rest of this chapter. To compute the quantisation of the action we are considering, we need to compute the kernels of

$$\begin{aligned} \mathcal{D}^+ &:= \mathcal{D}|_{\Omega^{0,\text{even}}(M)}; \\ \mathcal{D}^- &:= \mathcal{D}|_{\Omega^{0,\text{odd}}(M)}. \end{aligned}$$

This is not easy to do in general. But for  $n = 1$ , these kernels are computed in Section 11.6.

In our expression for the Dirac operator, we will use multi-indices

$$l = (l_1, \dots, l_q) \subset \{1, \dots, n\},$$

where  $q \in \{0, \dots, n\}$  and  $l_1 < \dots < l_q$ . We will write  $d\bar{z}^l := d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q}$ . If  $l = \emptyset$ , we set  $d\bar{z}^l := 1_M$ , the constant function 1 on  $M$ . Note that  $\{d\bar{z}^l\}_{l \subset \{1, \dots, n\}}$  is a  $C^\infty(M)$ -basis of  $\Omega^{0,*}(M; L)$ .

Given  $l \subset \{1, \dots, n\}$  and  $j \in \{1, \dots, n\}$ , we define

$$\varepsilon_{jl} := (-1)^{\#\{r \in \{1, \dots, q\}; l_r < j\}},$$

plus one if an even number of  $l_r$  is smaller than  $j$ , and minus one if the number of such  $l_r$  is odd. From the definition of the Dolbeault–Dirac operator one then deduces:

**Proposition 11.12.** *For all  $l \subset \{1, \dots, n\}$  and all  $f \in C^\infty(M)$ , we have*

$$\begin{aligned} \mathcal{D}(fd\bar{z}^l) &= \sum_{j \in l} \varepsilon_{jl} \left( -2 \frac{\partial f}{\partial \bar{z}_j} + (i\pi - 4\pi i p_j) f \right) d\bar{z}^{l \setminus \{j\}} \\ &\quad + \sum_{\substack{1 \leq j \leq n, \\ j \notin l}} \varepsilon_{jl} \left( \frac{\partial f}{\partial \bar{z}_j} + \frac{i\pi}{2} f \right) d\bar{z}^{l \cup \{j\}}. \end{aligned} \quad (11.13)$$

## 11.6 The case $n = 1$

We now consider the case where  $n = 1$ . That is,  $M = \mathbb{C}$  and  $\Gamma = \mathbb{Z} + i\mathbb{Z}$ . We can then explicitly compute the quantisation of the action of  $\Gamma$  on  $M$ . This will allow us to illustrate the Guillemin–Sternberg–Landsman conjecture by computing the four corners in diagram (3.16).

If  $n = 1$ , Proposition 11.12 reduces to

**Corollary 11.13.** *The Dirac operator on  $\mathbb{C}$ , coupled to  $L$ , is given by*

$$\mathcal{D}(f_1 + f_2 d\bar{z}) = \left( \frac{\partial f_1}{\partial \bar{z}} + \frac{i\pi}{2} f_1 \right) d\bar{z} - 2 \frac{\partial f_2}{\partial \bar{z}} + (i\pi - 4\pi i p) f_2.$$

That is to say, with respect to the  $C^\infty(M)$ -basis  $\{1_M, d\bar{z}\}$  of  $\Omega^{0,*}(M; L)$ , the Dirac operator  $\mathcal{D}$  has the matrix form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{D}^+ &= \frac{\partial}{\partial \bar{z}} + \frac{i\pi}{2}; \\ \mathcal{D}^- &= -2 \frac{\partial}{\partial \bar{z}} + i\pi - 4\pi i p. \end{aligned}$$

In this case, the kernels of  $\mathcal{D}^+$  and  $\mathcal{D}^-$  can be determined explicitly:

**Proposition 11.14.** *The kernel of  $\mathcal{D}^+$  consists of the sections  $s$  of  $L$  given by*

$$s(z) = e^{-i\pi \bar{z}/2} \varphi(z),$$



where  $\varphi$  is a holomorphic function.

The kernel of  $\mathcal{D}^-$  is isomorphic to the space of smooth sections  $t$  of  $L$  given by

$$t(z) = e^{i\pi z/2 + \pi|z|^2 - \pi z^2/2} \overline{\psi(z)},$$

where  $\psi$  is a holomorphic function.

The unitary dual of the group  $\mathbb{Z} + i\mathbb{Z} = \mathbb{Z}^2$  is the torus  $\mathbb{T}^2$ . Therefore, by Proposition 11.3, the quantisation of the action of  $\mathbb{Z} + i\mathbb{Z}$  on  $\mathbb{C}$  is the class in  $KK(\mathbb{C}, C^*(\mathbb{Z}^2))$  that corresponds to the class

$$\left[ \left( \ker \mathcal{D}_{(\alpha, \beta)}^+ \right)_{(\alpha, \beta) \in \mathbb{T}^2} \right] - \left[ \left( \ker \mathcal{D}_{(\alpha, \beta)}^- \right)_{(\alpha, \beta) \in \mathbb{T}^2} \right]$$

in  $K^0(\mathbb{T}^2)$ . It will turn out that the kernels of  $\mathcal{D}_{(\alpha, \beta)}^+$  and  $\mathcal{D}_{(\alpha, \beta)}^-$  indeed define vector bundles over  $\mathbb{T}^2$ . Let us compute these kernels.

**Proposition 11.15.** *Let  $\lambda, \mu \in \mathbb{R}$ . Define the section  $s_{\lambda\mu} \in \Gamma^\infty(M, L)$  by*

$$s_{\lambda\mu}(z) = e^{i\lambda z} e^{-\pi p} \sum_{k \in \mathbb{Z}} e^{-\pi k^2} e^{-k(\lambda + i\mu + 2\pi)} e^{2\pi i k z}.$$

Set  $\alpha := e^{i\lambda}$  and  $\beta := e^{i\mu}$ . Then  $\ker \mathcal{D}_{(\alpha, \beta)}^+ = \mathbb{C} s_{\lambda\mu}$ .

**Remark 11.16.** For all  $\lambda, \mu \in \mathbb{R}$ , we have

$$\begin{aligned} s_{\lambda+2\pi, \mu} &= e^{\lambda + i\mu + 3\pi} s_{\lambda\mu}; \\ s_{\lambda, \mu+2\pi} &= s_{\lambda\mu}. \end{aligned}$$

Hence the vector space  $\mathbb{C} s_{\lambda\mu} \subset \Gamma^\infty(M, L)$  is invariant under  $\lambda \mapsto \lambda + 2\pi$  and  $\mu \mapsto \mu + 2\pi$ . This is in agreement with the fact that  $\mathbb{C} s_{\lambda\mu}$  is the kernel of  $\mathcal{D}_{(e^{i\lambda}, e^{i\mu})}^+$ .

*Sketch of proof of Proposition 11.15.* Let  $\lambda, \mu \in \mathbb{R}$ , and  $s \in \Gamma^\infty(M, L) = C^\infty(\mathbb{C}, \mathbb{C})$ . Suppose  $s$  is in the kernel of  $\mathcal{D}_{(e^{i\lambda}, e^{i\mu})}^+$ . Let  $\varphi$  be the holomorphic function from Proposition 11.14, and write

$$\tilde{\varphi}(z) := e^{-i\lambda z} e^{-i\pi z/2} \varphi(z) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k z}$$

(note that for all  $z \in \mathbb{C}$ , one has  $\tilde{\varphi}(z+1) = \tilde{\varphi}(z)$ ). Then it follows from invariance of  $s$  under the action of the subgroup  $i\mathbb{Z}$  of  $\Gamma$  that  $a_k = e^{-\pi k^2} e^{-k(\lambda + i\mu + 2\pi)} a_0$ , which gives the desired result.  $\square$

**Proposition 11.17.** *The kernel of  $\mathcal{D}_{(\alpha, \beta)}^-$  is trivial for all  $(\alpha, \beta) \in \mathbb{T}^2$ .*

*Sketch of proof.* Let  $\lambda, \mu \in \mathbb{R}$  and let  $t d\bar{z} \in \Omega^{0,1}(M; L) = \Gamma^\infty(M, L) d\bar{z}$ . Suppose that  $t d\bar{z} \in \ker / D_{(e^{i\lambda}, e^{i\mu})}^-$ . Let  $\psi$  be the holomorphic function from Proposition 11.14, and write

$$\tilde{\psi}(z) := e^{\pi(\bar{z}^2 + i\bar{z})/2 - i\lambda \bar{z}} \overline{\psi(z)} = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \bar{z}}$$

(note that for all  $z \in \mathbb{C}$ , one has  $\tilde{\psi}(z+1) = \tilde{\psi}(z)$ ). Then it follows from invariance of  $t d\bar{z}$  under the action of the subgroup  $i\mathbb{Z}$  of  $\Gamma$  that  $c_k = e^{\pi k^2} e^{k(\lambda - i\mu - 2\pi)} c_0$ , which implies that  $c_0 = 0$ .  $\square$

We conclude:

**Proposition 11.18.** *The quantisation of the action of  $\mathbb{Z}^2$  on  $\mathbb{C}$  is the class in  $K^0(\mathbb{T}^2)$  defined by the vector bundle<sup>2</sup>*

$$(\mathbb{C}s_{\lambda\mu})_{(e^{i\lambda}, e^{i\mu}) \in \mathbb{T}^2} \rightarrow \mathbb{T}^2.$$

By Lemma 11.10, we now find that the reduction of the quantisation of the action of  $\mathbb{Z}^2$  on  $\mathbb{R}^2$  is the one-dimensional vector space  $\mathbb{C} \cdot s_{0,0} \subset \Gamma^\infty(M, L)$ , where

$$s_{0,0}(z) = e^{-\pi p} \sum_{k \in \mathbb{Z}} e^{-\pi k^2} e^{-2\pi k} e^{2\pi i k z}.$$

As we saw in Section 11.1, it follows from Proposition 10.1 that this is precisely the index of the Dolbeault–Dirac operator  $\bar{\partial}_{L/\mathbb{Z}^2} + \bar{\partial}_{L/\mathbb{Z}^2}^*$  on the torus  $\mathbb{T}^2$ , coupled to the line bundle  $L/\mathbb{Z}^2$  via the connection induced by  $\nabla$ . Schematically, we therefore have

$$\begin{array}{ccc} \mathbb{Z}^2 \circlearrowleft \mathbb{R}^2 & \xrightarrow{Q} & (\mathbb{C}s_{\lambda\mu})_{(e^{i\lambda}, e^{i\mu}) \in \mathbb{T}^2} \\ R_{\mathbb{Z}^2}^0 \downarrow & & \downarrow R_{\mathbb{Z}^2}^0 \\ \mathbb{T}^2 & \xrightarrow{Q} & \mathbb{C} \cdot s_{0,0}. \end{array} \quad (11.14)$$

Note that it is a coincidence that the two-torus appears twice in this diagram: in this example  $M/\Gamma = \mathbb{T}^2 = \hat{\Gamma}$ .

**Remark 11.19.** The fact that the geometric quantisation of the torus  $\mathbb{T}^2$  is one-dimensional can alternatively be deduced from the Atiyah–Singer index theorem for Dirac operators. Indeed, let  $\bar{\partial}_{L/\mathbb{Z}^2} + \bar{\partial}_{L/\mathbb{Z}^2}^*$  be the Dirac operator on the torus, coupled to the quotient line bundle  $L/\mathbb{Z}^2$ . Then by Atiyah–Singer, in the form stated for example in [27] on page 117, one has

$$\begin{aligned} Q(\mathbb{T}^2) &= \text{index}(\bar{\partial}_{L/\mathbb{Z}^2} + \bar{\partial}_{L/\mathbb{Z}^2}^*) = \int_{\mathbb{T}^2} e^{\text{ch}_1(L/\mathbb{Z}^2)} \\ &= \int_{\mathbb{T}^2} dp \wedge dq \\ &= 1, \end{aligned}$$

the symplectic volume of the torus, i.e. the volume determined by the Liouville measure.

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<sup>2</sup>By Remark 11.16, this is indeed a well-defined vector bundle.

## **Part IV**

# **Discrete series representations of semisimple groups**

In Part IV, we consider a cocompact Hamiltonian action of a semisimple Lie group  $G$  on a symplectic manifold  $(M, \omega)$ , and prove Theorem 6.13. The strategy of this proof is to deduce Theorem 6.13 from the (known) case of the action of a maximal compact subgroup  $K < G$  on the compact submanifold  $N := \Phi^{-1}(\mathfrak{k}^*)$  of  $M$ , with  $\Phi : M \rightarrow \mathfrak{g}^*$  the momentum map.

We will see in Chapter 12 that there are inverse constructions

$$\begin{aligned} \text{H-Cross}_K^G : G \circlearrowleft M &\rightsquigarrow K \circlearrowleft N := \Phi^{-1}(\mathfrak{k}^*); \\ \text{H-Ind}_K^G : K \circlearrowleft N &\rightsquigarrow G \circlearrowleft M := G \times_K N. \end{aligned}$$

These are called Hamiltonian cross-section and Hamiltonian induction, respectively. In Chapter 13, we define induction procedures for prequantisations, almost complex structures and  $\text{Spin}^c$ -structures, compatible with this Hamiltonian induction procedure.

The central result in Part IV is Theorem 14.5, which states that ‘quantisation commutes with induction’. Roughly speaking, this is expressed by the diagram

$$\begin{array}{ccc} (M = G \times_K N, \omega) & \xrightarrow{Q_G} & Q_G(M, \omega) \in K_0(C_r^*(G)) \\ \text{H-Ind}_K^G \uparrow & & \uparrow \text{D-Ind}_K^G \\ (N, \nu) & \xrightarrow{Q_K} & Q_K(N, \nu) \in R(K). \end{array}$$

Here  $R(K)$  is the representation ring of  $K$ ,  $K_0(C_r^*(G))$  is the  $K$ -theory of the reduced  $C^*$ -algebra of  $G$ , and  $\text{D-Ind}_K^G$  is the Dirac induction map (6.10). In Chapter 14, we tie the other chapters in Part IV together, by showing how Theorem 14.5 implies Theorem 6.13, and by sketching a proof of Theorem 14.5. The details of this proof are filled in in Chapter 15.

Our proof Theorem 14.5 is based on naturality of the assembly map for the inclusion of  $K$  into  $G$  (Theorem 9.1). In Chapter 15, we show that this naturality result is well-behaved with respect to the  $K$ -homology classes of the Dirac operators we use, thus proving Theorem 14.5.

Unless stated otherwise, we will use the notation and assumptions of Chapter 6. A large part of Part IV is about the relation between structures on the manifolds  $M$  and  $N$ . To avoid confusion, we use a superscript  $M$  or  $N$  to indicate if a given structure is defined on  $M$  or on  $N$ . In this way, we will have the momentum maps  $\Phi^M$  and  $\Phi^N$ , and the almost complex structures  $J^M$  and  $J^N$ , for example.

# Chapter 12

## Induction and cross-sections of Hamiltonian group actions

In this chapter, we explain the Hamiltonian induction and Hamiltonian cross-section constructions mentioned in the introduction to Part IV. We will see in Section 12.4 that they are each other's inverses. Our term 'Hamiltonian induction' is quite different from Guillemin and Sternberg's term 'symplectic induction' introduced in [29], Section 40.

Many results in this chapter are known for the case where the pair  $(G, K)$  is replaced by  $(K, T)$ . See for example [54, 63].

### 12.1 The tangent bundle to a fibred product

In our study of the manifold  $G \times_K N$ , we will use an explicit description of its tangent bundle, which we will now explain.

For this section, let  $G$  be any Lie group,  $H < G$  any closed subgroup, and  $N$  a left  $H$ -manifold. We consider the action of  $H$  on the product  $G \times N$  defined by

$$h \cdot (g, n) = (gh^{-1}, hn),$$

for all  $h \in H$ ,  $g \in G$  and  $n \in N$ . We denote the quotient of this action by  $G \times_H N$ , or by  $M$ . Because the action of  $H$  on  $G \times N$  is proper and free,  $M$  is a smooth manifold. We would like to describe the tangent bundle to  $M$  explicitly.

To this end, we endow the tangent bundle  $TH \cong H \times \mathfrak{h}$  with the group structure

$$(h, X)(h', X') := (hh', \text{Ad}(h)X' + X),$$

for  $h, h' \in H$  and  $X, X' \in \mathfrak{h}$ . This is a special case of the semidirect product group structure on a product  $V \rtimes H$ , where  $V$  is a representation space of  $H$ . We consider the action of the group  $TH$  on  $TG \times TN$  defined by

$$(h, X) \cdot (g, Y, v) := (gh^{-1}, \text{Ad}(h)Y - X, T_h h(v) + X_{hn}),$$

for  $h \in H$ ,  $X \in \mathfrak{h}$ ,  $(g, Y) \in G \times \mathfrak{g} \cong TG$ ,  $n \in N$  and  $v \in T_n N$ . Let  $TG \times_{TH} TN$  be the quotient of this action. It is a vector bundle over  $M$ , with projection map  $[g, X, v] \mapsto [g, n]$  (notation as above). We let  $G$  act on  $TG \times_{TH} TN$  by left multiplication on the first factor.

**Proposition 12.1.** *There is a  $G$ -equivariant isomorphism of vector bundles*

$$\Psi : TG \times_{TH} TN \rightarrow TM,$$

given by

$$\Psi[g, Y, v] = Tp(g, Y, v),$$

with  $p : G \times N \rightarrow M$  the quotient map.

*Proof.* Let us first show that  $\Psi$  is well-defined. Let  $g \in G$ ,  $Y \in \mathfrak{g}$ ,  $v \in T_n N$ ,  $h \in H$  and  $X \in \mathfrak{h}$  be given. Let  $\gamma$  be a curve in  $N$  with  $\gamma(0) = n$  and  $\gamma'(0) = v$ . Define the curve  $\delta$  in  $G \times N$  by

$$\delta(t) = (gh^{-1} \exp(t \operatorname{Ad}(h)Y) \exp(-tX), \exp(tX) \cdot h \cdot \gamma(t)).$$

Then

$$\delta'(0) = (gh^{-1}, \operatorname{Ad}(h)Y - X, T_n h(v) + X_{hn}) \in G \times \mathfrak{g} \times T_{hn} N.$$

Now since for all  $t$ ,

$$\begin{aligned} p \circ \delta(t) &= p(gh^{-1} \exp(t \operatorname{Ad}(h)Y) \exp(-tX), \exp(tX) \cdot h \cdot \gamma(t)) \\ &= p(g \exp(tY) h^{-1} \exp(-tX), \exp(tX) \cdot h \cdot \gamma(t)) \\ &= p(g \exp(tY), \gamma(t)), \end{aligned}$$

we have

$$\begin{aligned} Tp(gh^{-1}, \operatorname{Ad}(h)Y - X, T_n h(v) + X_{hn}) &= \left. \frac{d}{dt} \right|_{t=0} p \circ \delta(t) \\ &= \left. \frac{d}{dt} \right|_{t=0} p(g \exp(tY), \gamma(t)) \\ &= Tp(g, Y, v). \end{aligned}$$

So  $\Psi$  is indeed well-defined.

The map  $\Psi$  is a surjective vector bundle homomorphism because  $Tp : TG \times TN \rightarrow TM$  is. Because the bundles  $TM$  and  $TG \times_{TH} TN$  have the same rank, the map  $\Psi$  is therefore an isomorphism of vector bundles.  $\square$

Now suppose that there is an  $\operatorname{Ad}(H)$ -invariant linear subspace  $\mathfrak{p} \subset \mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  (such as in the case  $H = K$  we consider in the rest of Part IV). Then there is a possibly simpler description of  $TM$ , that we will also use later. Consider the action of  $H$  on the product  $G \times TN \times \mathfrak{p}$  given by

$$h \cdot (g, v, Y) = (gh^{-1}, T_n h(v), \operatorname{Ad}(h)Y),$$

and denote the quotient by  $G \times_H (TN \times \mathfrak{p})$ .

**Lemma 12.2.** *The map*

$$\Xi : TG \times_{TH} TN \rightarrow G \times_H (TN \times \mathfrak{p}),$$

given by

$$\Xi[g, Y, v] = [g, v + (Y_{\mathfrak{h}})_n, Y_{\mathfrak{p}}]$$

for all  $g \in G$ ,  $Y \in \mathfrak{g}$ ,  $n \in N$  and  $v \in T_n N$ , is a well-defined,  $G$ -equivariant isomorphism of vector bundles. Here  $Y_{\mathfrak{h}}$  and  $Y_{\mathfrak{p}}$  are the components of  $Y$  in  $\mathfrak{h}$  and  $\mathfrak{p}$  respectively, according to the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ .

Because of Proposition 12.1 and Lemma 12.2, we have  $TM \cong G \times_H (TN \times \mathfrak{p})$  as  $G$ -vector bundles.<sup>1</sup>

*Proof.* We first show that  $\Xi$  is well-defined. Indeed, for all  $g \in G$ ,  $Y \in \mathfrak{g}$ ,  $n \in N$  and  $v \in T_n N$ , and for all  $h \in H$  and  $X \in \mathfrak{h}$ , we have

$$\begin{aligned} \Xi[(h, X) \cdot (g, Y, v)] &= \\ [gh^{-1}, T_n h(v) + X_{hn} + ((\text{Ad}(h)Y - X)_{\mathfrak{h}})_{hn}, (\text{Ad}(h)Y - X)_{\mathfrak{p}}] &= \\ [gh^{-1}, T_n h(v) + (\text{Ad}(h)(Y_{\mathfrak{h}}))_{hn}, \text{Ad}(h)Y_{\mathfrak{p}}] \in G \times_H (TN \times \mathfrak{p}). \end{aligned} \quad (12.1)$$

Here we have used the fact that the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  is  $\text{Ad}(H)$ -invariant. Furthermore, we have

$$\begin{aligned} (\text{Ad}(h)(Y_{\mathfrak{h}}))_{hn} &= \left. \frac{d}{dt} \right|_{t=0} \exp(t \text{Ad}(h)Y_{\mathfrak{h}})hn \\ &= \left. \frac{d}{dt} \right|_{t=0} h \exp(tY_{\mathfrak{h}})n \\ &= T_n h(Y_{\mathfrak{h}})_n. \end{aligned}$$

Hence (12.1) equals

$$[h \cdot (g, v + (Y_{\mathfrak{h}})_n, Y_{\mathfrak{p}})] = [g, v + (Y_{\mathfrak{h}})_n, Y_{\mathfrak{p}}] = \Xi[g, Y, v],$$

which shows that  $\Xi$  is well-defined.

It is obvious that  $\Xi$  is fibrewise linear. Let us prove that it is fibrewise injective: with notation as above, suppose that

$$\Xi[g, Y, v] = [g, v + (Y_{\mathfrak{h}})_n, Y_{\mathfrak{p}}] = 0.$$

That is,  $Y \in \mathfrak{h}$  and  $v = -(Y_{\mathfrak{h}})_n$ . And therefore,

$$[g, Y, v] = [(e, -Y) \cdot (g, 0, 0)] = [g, 0, 0],$$

and  $\Xi$  is fibrewise injective. Hence, because  $\Xi$  is a map between vector bundles of the same rank, it is a fibrewise linear isomorphism.

Finally, the isomorphism  $\Xi$  is  $G$ -equivariant because on both sides,  $G$  acts by left multiplication on the first factor.  $\square$

In Chapter 13, we will use the following version of Proposition 12.1 and Lemma 12.2.

**Corollary 12.3.** *In the situation of Lemma 12.2, there is an isomorphism of  $G$ -vector bundles*

$$TM \cong (p_{G/H}^* T(G/H)) \oplus (G \times_H TN),$$

where  $p_{G/H} : M \rightarrow G/H$  is the natural projection.

*Proof.* The claim follows from Proposition 12.1, Lemma 12.2, and the fact that

$$T(G/H) \cong G \times_H \mathfrak{p},$$

where  $H$  acts on  $\mathfrak{p}$  via  $\text{Ad}$ .  $\square$

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<sup>1</sup>A version of this fact is used without a proof in [6] on page 503.

## 12.2 Hamiltonian induction

We return to the standard situation in Part IV, where  $G$  is a semisimple group, and  $K < G$  is a maximal compact subgroup.

### The symplectic manifold

Let  $(N, \nu)$  be a symplectic manifold on which  $K$  acts in Hamiltonian fashion, with momentum map  $\Phi^N : N \rightarrow \mathfrak{k}^*$ . Suppose that the image of  $\Phi^N$  lies in the set  $\mathfrak{k}_{\text{se}}^*$ , defined in (6.19). As in Section 12.1, we consider the fibred product  $M = G \times_K N$ , equipped with the action of  $G$  induced by left multiplication on the first factor. As a consequence of Proposition 12.1 and Lemma 12.2, we have for all  $n \in N$ ,

$$T_{[e,n]}M \cong T_nN \oplus \mathfrak{p}.$$

We define a two-form  $\omega$  on  $M$  by requiring that it is  $G$ -invariant, and that for all  $X, Y \in \mathfrak{p}$ ,  $n \in N$  and  $v, w \in T_nN$ ,

$$\omega_{[e,n]}(v + X, w + Y) := \nu_n(v, w) - \langle \Phi^N(n), [X, Y] \rangle. \quad (12.2)$$

Note that  $[X, Y] \in \mathfrak{k}$  for all  $X, Y \in \mathfrak{p}$ , so the pairing in the second term is well-defined. We claim that  $\omega$  is a symplectic form. This is analogous to formula (7.4) from [63].

**Proposition 12.4.** *The form  $\omega$  is symplectic.*

*Proof.* The form  $\omega$  is closed, because it is the curvature form of a connection on a line bundle over  $M$ . This will be proved in Section 13.1.

Next, we show that  $\omega$  is nondegenerate. By  $G$ -invariance of  $\omega$ , it is enough to prove this at points of the form  $[e, n]$ , with  $n \in N$ . Let  $v \in T_nN$  and  $X \in \mathfrak{p}$  be given, such that for all  $w \in T_nN$  and  $Y \in \mathfrak{p}$ , we have

$$\omega_{[e,n]}(v + X, w + Y) = 0. \quad (12.3)$$

Then in particular,

$$\omega_{[e,n]}(v + X, w) = \nu_n(v, w) = 0$$

for all such  $w$ , and hence  $v = 0$  by nondegeneracy of  $\nu$ .

On the other hand, we have

$$0 = \omega_{[e,n]}(v + X, Y)$$

for all  $Y \in \mathfrak{p}$ , which equals

$$-\langle \Phi^N(n), [X, Y] \rangle = \langle \text{ad}^*(X) \Phi^N(n), Y \rangle = \langle X_{\Phi^N(n)}, Y \rangle.$$

Analogously, for  $Z \in \mathfrak{k}$  we have

$$\langle X_{\Phi^N(n)}, Z \rangle = -\langle \Phi^N(n), [X, Z] \rangle,$$

which also equals zero, since  $[X, Z] \in \mathfrak{p}$  and  $\Phi^N(n) \in \mathfrak{k}^* \cong \mathfrak{p}^0$ . Therefore,  $X_{\Phi^N(n)} = 0$ , which by Lemma 6.11 implies that  $X = 0$ , since  $\Phi^N(N) \subset \mathfrak{k}_{\text{se}}^*$ . We conclude that  $\omega_{[e,n]}$  is indeed nondegenerate.  $\square$



### The momentum map

Next, consider the map  $\Phi^M : M \rightarrow \mathfrak{g}^*$  given by

$$\Phi^M[g, n] = \text{Ad}^*(g)\Phi^N(n). \quad (12.4)$$

This map is well-defined by  $K$ -equivariance of  $\Phi^N$ . Furthermore, it is obviously  $G$ -equivariant, and its image lies in  $\mathfrak{g}_{\text{se}}^*$ .

**Proposition 12.5.** *The map  $\Phi^M$  is a momentum map for the action of  $G$  on  $M$ .*

*Proof.* We first prove the defining property of momentum maps,

$$d\Phi_X^M = -X_M \lrcorner \omega \quad (12.5)$$

for all  $X \in \mathfrak{g}$ , at points of the form  $[e, n]$ , with  $n \in N$ . To this end, we compute the tangent map  $T_{[e, n]}\Phi^M$  in the following way. Let  $v \in T_n N$  and  $Y \in \mathfrak{p}$  be given. Let  $\gamma$  be a curve in  $N$  such that  $\gamma(0) = n$  and  $\gamma'(0) = v$ . Then

$$\begin{aligned} T_{[e, n]}\Phi^M(v + Y) &= \left. \frac{d}{dt} \right|_{t=0} \Phi^M[\exp(tY), \gamma(t)] \\ &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}^*(\exp tY)\Phi^N(\gamma(t)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \Phi^N(\gamma(t)) + \left. \frac{d}{dt} \right|_{t=0} \text{Ad}^*(\exp tY)\Phi^N(n) \\ &= T_n\Phi^N(v) + \text{ad}^*(Y)\Phi^N(n). \end{aligned}$$

Now let  $X \in \mathfrak{g}$  and let  $Y, v$  be as before. Write  $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ , with  $X_{\mathfrak{k}} \in \mathfrak{k}$  and  $X_{\mathfrak{p}} \in \mathfrak{p}$ . Then

$$\begin{aligned} \langle d_{[e, n]}\Phi_X^M, v + Y \rangle &= \langle T_{[e, n]}\Phi^M(v + Y), X \rangle \\ &= \langle T_n\Phi^N(v), X \rangle + \langle \text{ad}^*(Y)\Phi^N(n), X \rangle \\ &= \langle T_n\Phi^N(v), X_{\mathfrak{k}} \rangle + \langle \Phi^N(n), [X, Y]_{\mathfrak{k}} \rangle. \end{aligned} \quad (12.6)$$

By the defining property of  $\Phi^N$ , and because  $[X, Y]_{\mathfrak{k}} = [X_{\mathfrak{p}}, Y]$ , the expression (12.6) equals

$$-v_n((X_{\mathfrak{k}})_n, v) + \langle \Phi^N(n), [X_{\mathfrak{p}}, Y] \rangle = -\omega_{[e, n]}((X_{\mathfrak{k}})_n + X_{\mathfrak{p}}, v + Y).$$

By Lemma 12.6 below, we have  $X_{[e, n]} = (X_{\mathfrak{k}})_n + X_{\mathfrak{p}}$ , which yields equality (12.5) at the point  $[e, n]$ .

To prove (12.5) on all of  $M$ , we note that on both sides of this equation, pulling back along an element  $g \in G$  amounts to replacing  $X$  by  $\text{Ad}(g)X$ , as one can compute. Therefore, equality (12.5) at points of the form  $[e, n]$  implies the general case.  $\square$

In the proof of Proposition 12.5, we used:

**Lemma 12.6.** *With notation as before, we have*

$$X_{[e, n]} = (X_{\mathfrak{k}})_n + X_{\mathfrak{p}}$$

in  $T_n N \oplus \mathfrak{p} \cong T_{[e, n]}M$ .

*Proof.* Using the isomorphisms  $TM \cong TG \times_{TK} TN$  and  $TG \times_{TK} TN \cong G \times_K (TN \times \mathfrak{p})$  from Proposition 12.1 and Lemma 12.2, we compute

$$\begin{aligned} X_{[e,n]} &= \left. \frac{d}{dt} \right|_{t=0} [\exp tX, n] \in T_{[e,n]}M \\ &\mapsto [e, X, 0] \in TG \times_{TK} TN \\ &= [e, X_{\mathfrak{p}}, (X_{\mathfrak{k}})_n] \\ &\mapsto (X_{\mathfrak{k}})_n + X_{\mathfrak{p}} \in T_n N \oplus \mathfrak{p}. \end{aligned}$$

□

**Definition 12.7.** The *Hamiltonian induction* of the Hamiltonian action of  $K$  on  $(N, \nu)$  is the Hamiltonian action of  $G$  on  $(M, \omega)$ :

$$\text{H-Ind}_K^G(N, \nu, \Phi^N) := (M, \omega, \Phi^M).$$

**Example 12.8.** Let  $\xi \in \mathfrak{k}^* \setminus \text{ncw}$  be given, and consider the coadjoint orbit  $N := K \cdot \xi \subset \mathfrak{k}^*$ . The Hamiltonian induction of the coadjoint action of  $K$  on  $N$  is the coadjoint action of  $G$  on the coadjoint orbit  $M := G \cdot \xi$ , including the natural symplectic forms and momentum maps. Indeed, the map

$$G \cdot \xi \rightarrow G \times_K N$$

given by  $g \cdot \xi \mapsto [g, \xi]$  is a symplectomorphism.

## 12.3 Hamiltonian cross-sections

We now turn to the inverse construction to Hamiltonian induction, namely the *Hamiltonian cross-section*. In this case, we start with a Hamiltonian  $G$ -manifold  $(M, \omega)$ , with momentum map  $\Phi^M$ . Such a cross-section will indeed be symplectic and carry a Hamiltonian  $K$ -action, under the assumption that the image of  $\Phi^M$  is contained in  $\mathfrak{g}_{\text{se}}^*$ . A Hamiltonian cross-section is a kind of double restriction: it is both a restriction to a subgroup of  $G$  and a restriction to a submanifold of  $M$ .

Most of this section is based on the proof of the symplectic cross-section theorem in Lerman et al. [54].

As before, we identify  $\mathfrak{k}^*$  with the subspace  $\mathfrak{p}^0$  of  $\mathfrak{g}^*$ . The main result of this section is:

**Proposition 12.9.** *If  $\Phi^M(M) \subset \mathfrak{g}_{\text{se}}^*$ , then  $N := (\Phi^M)^{-1}(\mathfrak{k}^*)$  is a  $K$ -invariant symplectic submanifold of  $M$ , and  $\Phi^N := \Phi^M|_N$  is a momentum map for the action of  $K$  on  $N$ .*

We denote the restricted symplectic form  $\omega|_N$  by  $\nu$ .

**Definition 12.10.** The *Hamiltonian cross-section* of the Hamiltonian action of  $G$  on  $(M, \omega)$  is the Hamiltonian action of  $K$  on  $(N, \nu)$ :

$$\text{H-Cross}_K^G(M, \omega, \Phi^M) := (N, \nu, \Phi^N).$$

In Proposition 12.15, we will see that  $M \cong G \times_K N$ , so that  $M/G$  is compact if and only if  $N$  is.

To prove Proposition 12.9, we have to show that  $N$  is a smooth submanifold of  $M$ , and that the restricted form  $\omega|_N$  is symplectic. Then the submanifold  $N$  is  $K$ -invariant by  $K$ -equivariance of  $\Phi^M$ , and the fact that  $\Phi^N$  is a momentum map is easily verified. We begin with some preparatory lemmas, based on the proof of the symplectic cross-section theorem mentioned above.

For the remainder of this section, let  $m \in M$  be given, and write  $\xi := \Phi^M(m)$ .

**Lemma 12.11.** *The linear map*

$$\psi : T_m(G \cdot m) \rightarrow T_\xi(G \cdot \xi)$$

given by

$$\psi(X_m) = X_\xi$$

for  $X \in \mathfrak{g}$ , is symplectic, in the sense that for all  $X, Y \in \mathfrak{g}$ ,

$$\omega_m(X_m, Y_m) = -\langle \xi, [X, Y] \rangle.$$

*Proof.* First note that  $\psi$  is well-defined because by equivariance of  $\Phi^M$ , we have  $\mathfrak{g}_m \subset \mathfrak{g}_\xi$ .

Furthermore, by the properties of  $\Phi^M$  we have

$$\begin{aligned} \omega_m(X_m, Y_m) &= -\langle d_m \Phi_X^M, Y_m \rangle \\ &= -\langle T_m \Phi^M(Y_m), X \rangle \\ &= -\left. \frac{d}{dt} \right|_{t=0} \langle \Phi^M(\exp(tY)m), X \rangle \\ &= -\left. \frac{d}{dt} \right|_{t=0} \langle \text{Ad}^*(\exp tY) \Phi^M(m), X \rangle \\ &= -\langle \text{ad}^*(Y) \xi, X \rangle \\ &= -\langle \xi, [X, Y] \rangle. \end{aligned}$$

□

**Lemma 12.12.** *We have the following inclusions of subspaces of  $\mathfrak{g}^*$ :*

$$\mathfrak{g}_\xi^0 \subset T_m \Phi^M(T_m M) \subset \mathfrak{g}_m^0.$$

*Proof.* The second inclusion is the easiest one to prove. Indeed, let  $v \in T_m M$  and  $X \in \mathfrak{g}_m$  be given. Then by definition of momentum maps,

$$\langle T_m \Phi^M(v), X \rangle = \langle d_m \Phi_X^M, v \rangle = -\omega(X_m, v) = 0,$$

since  $X_m = 0$ .

To prove the first inclusion, we consider the maps

$$\mathfrak{g}_\xi^0 \cong (\mathfrak{g}/\mathfrak{g}_\xi)^* \cong T_\xi^*(G \cdot \xi) \xleftarrow{\cong} T_\xi(G \cdot \xi) \xleftarrow{\psi} T_m(G \cdot m) \hookrightarrow T_m M.$$

Here ‘#’ denotes the isomorphism induced by the standard symplectic form on  $G \cdot \xi$  (see Example 2.13).

Let  $\eta \in \mathfrak{g}_\xi^0$  be given, and choose  $v \in T_m(G \cdot m)$  such that the images of  $v$  and  $\eta$  in  $T_\xi(G \cdot \xi)$  under the maps above coincide. (Note that such a  $v$  exists since  $\psi$  is surjective.) We claim that  $T_m\Phi^M(v) = \eta$ . Indeed, write  $v = X_m$  for an  $X \in \mathfrak{g}$ . Then for all  $Y \in \mathfrak{g}$ ,

$$\langle \eta, Y \rangle = \langle \xi, [X, Y] \rangle = -\omega_m(X_m, Y_m)$$

by the definition of the map #, and by Lemma 12.11. By definition of  $\Phi^M$ , the latter expression equals  $\langle T_m\Phi^M(v), Y \rangle$ , which proves the claim.  $\square$

**Lemma 12.13.** *If  $m \in N \subset M$ , then the subspace*

$$\mathfrak{p} \cdot m := \{X_m; X \in \mathfrak{p}\} \subset T_mM$$

*is symplectic.*

*Proof.* Step 1: we have

$$T_\xi(G \cdot \xi) \cong \mathfrak{g} \cdot \xi = (\mathfrak{k} + \mathfrak{p}) \cdot \xi = T_\xi(K \cdot \xi) + \mathfrak{p} \cdot \xi.$$

Step 2: the subspace  $\mathfrak{p} \cdot \xi \subset T_\xi(G \cdot \xi)$  is symplectic.

Indeed, by Step 1 and Lemma 12.14 below, it is enough to prove that  $\mathfrak{p} \cdot \xi$  and  $T_\xi(K \cdot \xi)$  are symplectically orthogonal. Let  $X \in \mathfrak{k}$  and  $Y \in \mathfrak{p}$  be given. Because  $m \in N$ , we have  $\xi \in \mathfrak{k}^*$ , and also  $\text{ad}^*(X)\xi \in \mathfrak{k}^* \cong \mathfrak{p}^0$ . Hence

$$\langle \xi, [X, Y] \rangle = -\langle \text{ad}^*(X)\xi, Y \rangle = 0.$$

Step 3: the subspace  $\mathfrak{p} \cdot m \subset T_mM$  is symplectic.

Indeed, let a nonzero  $X \in \mathfrak{p}$  be given. We are looking for a  $Y \in \mathfrak{p}$  such that  $\omega_m(X_m, Y_m) \neq 0$ . Note that by Lemma 6.11, we have  $\text{ad}^*(X)\xi = X_\xi \neq 0$ . So by Step 2, there is a  $Y \in \mathfrak{p}$  for which  $\langle \xi, [X, Y] \rangle \neq 0$ . Hence by Lemma 12.11,

$$\omega_m(X_m, Y_m) = -\langle \xi, [X, Y] \rangle \neq 0.$$

$\square$

In Step 2 of the proof of Lemma 12.13, we used

**Lemma 12.14.** *Let  $(W, \sigma)$  be a symplectic vector space, and let  $U, V \subset W$  be linear subspaces. Suppose that  $W = U + V$ , and that  $U$  and  $V$  are symplectically orthogonal. Then  $U$  and  $V$  are symplectic subspaces.*

*Proof.* We prove the claim for  $U$ . Let  $u \in U \setminus \{0\}$  be given. Choose  $w \in W$  such that  $\sigma(u, w) \neq 0$ . Since  $W = U + V$ , there are  $u' \in U$  and  $v \in V$  such that  $w = u' + v$ . For such  $u'$ , we have

$$\sigma(u, u') = \sigma(u, w) \neq 0.$$

$\square$

After these preparations, we are ready to prove Proposition 12.9.

*Proof of Proposition 12.9.* We first show that  $N$  is smooth. This is true if  $\Phi^M$  satisfies the transversality condition that for all  $n \in N$ , with  $\eta := \Phi^M(n)$ , we have

$$T_\eta \mathfrak{g}^* = T_\eta \mathfrak{k}^* + T_n \Phi^M(T_n M).$$

(See e.g. [35], Chapter 1, Theorem 3.3.) By Lemma 12.12, we have  $\mathfrak{g}_\eta^0 \subset T_n \Phi^M(T_n M)$ , and by Lemma 6.11, we have  $\mathfrak{g}_\eta \cap \mathfrak{p} = \{0\}$ . Now, using the fact that  $V^0 + W^0 = (V \cap W)^0$  for two linear subspaces  $V$  and  $W$  of a vector space, we see that

$$T_\eta \mathfrak{k}^* + T_n \Phi^M(T_n M) \supset \mathfrak{p}^0 + \mathfrak{g}_\eta^0 = (\mathfrak{p} \cap \mathfrak{g}_\eta)^0 = \{0\}^0 = \mathfrak{g}^*.$$

This shows that  $N$  is indeed smooth.

Next, we prove that  $\omega|_N$  is a symplectic form. It is closed because  $\omega$  is, so it remains to show that it is nondegenerate. Let  $n \in N$  be given. By Lemma 12.14, it is enough to show that  $T_n M = T_n N + \mathfrak{p} \cdot n$ , and that  $T_n N$  and  $\mathfrak{p} \cdot n$  are symplectically orthogonal.

We prove that  $T_n M = T_n N \oplus \mathfrak{p} \cdot n$ , by first noting that

$$\dim N = \dim M - \dim \mathfrak{g}^* + \dim \mathfrak{k}^* = \dim M - \dim \mathfrak{p}.$$

Because  $\mathfrak{g}_n \subset \mathfrak{g}_{\Phi^M(n)}$ , and  $\mathfrak{g}_{\Phi^M(n)} \cap \mathfrak{p} = \{0\}$  by Lemma 6.11, we have  $\dim \mathfrak{p} = \dim(\mathfrak{p} \cdot n)$ , and

$$\dim T_n M = \dim T_n N + \dim(\mathfrak{p} \cdot n).$$

It is therefore enough to prove that  $T_n N \cap \mathfrak{p} \cdot n = \{0\}$ . To this end, let  $X \in \mathfrak{p}$  be given, and suppose  $X_n \in T_n N$ . That is,  $T_n \Phi^M(X_n) \in \mathfrak{k}^*$ , which is to say that for all  $Y \in \mathfrak{p}$ ,

$$\omega_n(X_n, Y_n) = -\langle T_n \Phi^M(X_n), Y \rangle = 0.$$

By Lemma 12.13, it follows that  $X_n = 0$ , so that indeed  $T_n N \cap \mathfrak{p} \cdot n = \{0\}$ .

Finally, we show that for all  $v \in T_n N$  and  $X \in \mathfrak{p}$ , we have  $\omega_n(v, X_n) = 0$ . Indeed, for such  $v$  and  $X$ , we have  $T_n \Phi^M(v) \in \mathfrak{k}^* \cong \mathfrak{p}^0$ , so

$$\omega_n(v, X_n) = \langle T_n \Phi^M(v), X \rangle = 0.$$

□

## 12.4 Hamiltonian induction and taking Hamiltonian cross-sections are mutually inverse

Let us prove the statement in the title of this section. One side of it (Proposition 12.15) will be used in the proof of Theorem 6.13 in Section 14.3. We will not use the other side (Proposition 12.16).

## Induction of a cross-section

First, we have

**Proposition 12.15.** *Let  $(M, \omega, \Phi^M)$  and  $(N, \nu, \Phi^N) := \text{H-Cross}_K^G(M, \omega, \Phi^M)$  be as in Section 12.3. Consider the manifold  $\tilde{M} := G \times_K N$ , with symplectic form  $\tilde{\omega}$  equal to the form  $\omega$  in (12.2). Define the map  $\widetilde{\Phi^M}$  as the map  $\Phi^M$  in (12.4). Then the map*

$$\varphi : \tilde{M} \rightarrow M$$

given by

$$\varphi[g, n] = g \cdot n$$

is a well-defined,  $G$ -equivariant symplectomorphism, and  $\varphi^* \Phi^M = \widetilde{\Phi^M}$ .

Put differently,  $\text{H-Ind}_K^G \circ \text{H-Cross}_K^G$  is the identity, modulo equivariant symplectomorphisms that intertwine the momentum maps.

It follows from this proposition that  $M/G = N/K$ , so that  $M/G$  is compact if and only if  $N$  is compact.

*Proof.* The statement about the momentum maps follows from  $G$ -equivariance of  $\Phi^M$ .

The map  $\varphi$  is well-defined by definition of the action of  $K$  on  $G \times N$ . It is obviously  $G$ -equivariant. Furthermore,  $\varphi$  is smooth because the action of  $G$  on  $M$  is smooth (this was a tacit assumption), and by definition of the smooth structure on the quotient  $G \times_K N$ .

To prove injectivity of  $\varphi$ , let  $g, g' \in G$  and  $n, n' \in N$  be given, and suppose that  $g \cdot n = g' \cdot n'$ . Because  $\Phi^M(N) \subset \mathfrak{k}_{\text{se}}^*$ , there are  $k, k' \in K$  and  $\xi, \xi' \in \mathfrak{t}_+^* \setminus \text{ncw}$  such that

$$\begin{aligned} \Phi^M(n) &= k \cdot \xi; \\ \Phi^M(n') &= k' \cdot \xi'. \end{aligned}$$

Then by equivariance of  $\Phi^M$ , we have  $gk \cdot \xi = g'k' \cdot \xi'$ . Because  $\mathfrak{t}_+^* \setminus \text{ncw}$  is a fundamental domain for the coadjoint action of  $G$  on  $\mathfrak{g}_{\text{se}}^*$ , we must have  $\xi = \xi'$ , and

$$k'^{-1}g'^{-1}gk \in G_\xi \subset K.$$

So  $k'' := g'^{-1}g \in K$ . Hence

$$g'k''n = g \cdot n = g' \cdot n',$$

and  $k'' \cdot n = n'$ . We conclude that

$$[g', n'] = [gk''^{-1}, k'' \cdot n] = [g, n],$$

and  $\varphi$  is injective.

To prove surjectivity of  $\varphi$ , let  $m \in M$  be given. Since  $\Phi^M(m) \in \mathfrak{g}_{\text{se}}^*$ , there are  $g \in G$  and  $\xi \in \mathfrak{t}_+^* \setminus \text{ncw}$  such that  $\Phi^M(m) = g \cdot \xi$ . Set  $n := g^{-1}m$ . Then  $\Phi^M(n) = \xi \in \mathfrak{k}^*$ , so  $n \in N$ , and  $\varphi[g, n] = m$ .

Next, we show that the inverse of  $\varphi$  is smooth. We prove this using the inverse function theorem: smoothness of  $\varphi^{-1}$  follows from the fact that the tangent map  $T\varphi$  is invertible. Or, equivalently, from the fact that the map  $\widetilde{T\varphi}$ , defined by the following diagram, is invertible.

$$\begin{array}{ccc} T(G \times_K N) & \xrightarrow{T\varphi} & TM \\ \Psi \uparrow \cong & \nearrow \widetilde{T\varphi} & \\ TG \times_{TK} TN & & \end{array}$$

Here  $\Psi$  is the isomorphism from Proposition 12.1. Explicitly, the map  $\widetilde{T\varphi}$  is given by

$$\begin{aligned} \widetilde{T\varphi}[g, X, v] &= T\varphi \circ Tp(g, X, v) \\ &= T\alpha(g, X, v), \end{aligned}$$

for all  $g \in G$ ,  $X \in \mathfrak{g}$  and  $v \in T_n N$ , with  $\alpha : G \times N \rightarrow M$  the action map. Let  $\gamma$  be a curve in  $N$  with  $\gamma(0) = n$  and  $\gamma'(0) = v$ . Then we find that

$$\begin{aligned} T\alpha(g, X, v) &= \left. \frac{d}{dt} \right|_{t=0} \exp(tX)g \cdot \gamma(t) \\ &= X_{gn} + T_n g(v). \end{aligned} \tag{12.7}$$

Because the vector bundles  $TG \times_{TK} TN$  and  $TM$  have the same rank, it is enough to show that  $\widetilde{T\varphi}$  is surjective. To this end, let  $m \in M$  and  $w \in T_m M$  be given. Since  $\varphi$  is surjective, there are  $g \in G$  and  $n \in N$  such that  $m = g \cdot n$ . Furthermore, we have

$$T_n M = T_n N + \mathfrak{g} \cdot n.$$

Indeed, in our situation we even have  $T_n M = T_n N \oplus \mathfrak{p} \cdot n$  (see the proof of Proposition 12.9). Hence

$$T_m M = T_n g(T_n M) = T_n g(T_n N + \mathfrak{g} \cdot n).$$

Therefore, there are  $v \in T_n N$  and  $X \in \mathfrak{g}$  such that

$$\begin{aligned} w &= T_n g(v + X_n) \\ &= T_n g(v) + (\text{Ad}(g)X)_{g \cdot n} \\ &= \widetilde{T\varphi}[g, \text{Ad}(g)X, v], \end{aligned}$$

by (12.7). This shows that  $\widetilde{T\varphi}$  is indeed surjective.

Finally, we prove that  $\varphi$  is a symplectomorphism. Let  $n \in N$ ,  $v, w \in T_n N$  and  $X, Y \in \mathfrak{p}$  be given. We will show that

$$\omega_n(T_{[e,n]}\varphi(v+X), T_{[e,n]}\varphi(w+Y)) = \omega_n(v, w) - \langle \Phi^M(n), [X, Y] \rangle.$$

By  $G$ -invariance of the symplectic forms  $\omega$  and  $\tilde{\omega}$ , this implies that  $\varphi$  is a symplectomorphism on all of  $\tilde{M}$ .

Similarly to (12.7), we find that  $T_{[e,n]}\varphi(v+X) = v + X_n$ . Therefore,

$$\begin{aligned} \omega_n(T_{[e,n]}\varphi(v+X), T_{[e,n]}\varphi(w+Y)) &= \omega_n(v + X_n, w + Y_n) \\ &= \omega_n(v, w) + \omega_n(X_n, Y_n), \end{aligned} \tag{12.8}$$

since  $T_n N$  and  $\mathfrak{p} \cdot n$  are symplectically orthogonal (see the end of the proof of Proposition 12.9). Now applying Lemma 12.11 to the first term in (12.8) gives the desired result.  $\square$

## Cross-section of an induction

Conversely to Proposition 12.15, we have:

**Proposition 12.16.** *Let  $(N, \nu, \Phi^N)$  and  $(M, \omega, \Phi^M) := \text{H-Ind}(N, \nu, \Phi^N)$  be as in Section 12.2. Suppose  $\Phi^N(N) \subset \mathfrak{k}_{\text{se}}^*$ . Then*

$$(N, \nu) \cong \left( (\Phi^M)^{-1}(\mathfrak{k}^*), \omega|_{(\Phi^M)^{-1}(\mathfrak{k}^*)} \right),$$

and this isomorphism intertwines the momentum maps  $\Phi^N$  and  $\Phi^M$ .

In other words,  $\text{H-Cross}_K^G \circ \text{H-Ind}_K^G$  is the identity, modulo equivariant symplectomorphisms that intertwine the momentum maps.

*Proof.* We claim that

$$(\Phi^M)^{-1}(\mathfrak{k}^*) = \{[e, n]; n \in N\} =: \tilde{N}. \quad (12.9)$$

The map  $n \mapsto [e, n]$  is a diffeomorphism from  $N$  to  $\tilde{N}$ . It is clear that this diffeomorphism is  $K$ -equivariant, and intertwines the momentum maps  $\Phi^N$  and  $\Phi^M$ .

To prove that  $(\Phi^M)^{-1}(\mathfrak{k}^*) = \tilde{N}$ , let  $[g, n] \in M$  be given, and suppose  $\Phi^M[g, n] = g \cdot \Phi^N(n) \in \mathfrak{k}^*$ . Because  $\Phi^N(N) \subset \mathfrak{k}_{\text{se}}^*$ , we have

$$g \cdot \Phi^N(n) \in (G \cdot \mathfrak{k}_{\text{se}}^*) \cap \mathfrak{k}^* = \mathfrak{k}_{\text{se}}^*.$$

So there are  $k, k' \in K$  and  $\xi, \xi' \in \mathfrak{t}_+^* \setminus \text{ncw}$  such that

$$\begin{aligned} \Phi^N(n) &= k \cdot \xi; \\ g \cdot \Phi^N(n) &= k' \cdot \xi'. \end{aligned}$$

Hence  $gk \cdot \xi = k' \cdot \xi'$ , and since  $\mathfrak{t}_+^* \setminus \text{ncw}$  is a fundamental domain for the coadjoint action of  $G$  on  $\mathfrak{g}_{\text{se}}^*$ , we have  $\xi' = \xi$ . So

$$k'^{-1}gk \in G_\xi \subset K,$$

and hence  $g \in K$ . We conclude that  $[g, n] = [e, g^{-1}n]$ , which proves (12.9) (the inclusion  $\tilde{N} \subset (\Phi^M)^{-1}(\mathfrak{k}^*)$  follows from the definition of  $\Phi^M$ ).

For each  $n \in N$ , the natural isomorphism  $\nu \mapsto [e, 0, \nu]$  from  $T_n N$  to  $T_{[e, n]} \tilde{N}$  intertwines the respective symplectic forms, by definition of those forms.  $\square$



# Chapter 13

## Induction of prequantisations and $\text{Spin}^c$ -structures

We extend the induction procedure of Chapter 12 to prequantisations and to  $\text{Spin}^c$ -structures, used to define quantisation. For prequantisations, it is possible to define restriction to a Hamiltonian cross-section in a suitable way. For our purposes, it is not necessary to restrict  $\text{Spin}^c$ -structures.

### 13.1 Prequantisations

Since we are interested in quantising Hamiltonian actions, let us look at induction of prequantum line bundles, and at restriction to Hamiltonian cross-sections.

#### Restriction to Hamiltonian cross-sections

The easy part is restriction. Indeed, let  $(M, \omega)$  be a Hamiltonian  $G$ -manifold, let  $\Phi^M$  be a momentum map with  $\Phi^M(M) \subset \mathfrak{g}_{\text{se}}^*$ , and let  $(N, \nu, \Phi^N)$  be the Hamiltonian cross-section of this action. Now let  $L^\omega \rightarrow M$  be a prequantum line bundle, let  $(-, -)_{L^\omega}$  be a  $G$ -invariant Hermitian metric on  $L^\omega$ , and let  $\nabla^M$  be a  $G$ -equivariant Hermitian connection on  $L^\omega$  with curvature  $2\pi i \omega$ . Let  $\nabla^N$  be the connection on  $L^\nu := L^\omega|_N$  defined as the pullback of  $\nabla^M$  along the inclusion map  $N \hookrightarrow M$ . It is given by

$$\nabla^N(s|_N) = (\nabla^M s)|_N,$$

for all sections  $s \in \Gamma^\infty(L^\omega)$ . This is indeed a connection, with curvature

$$R_{\nabla^N} = R_{\nabla^M}|_N = 2\pi i \omega|_N = 2\pi i \nu.$$

Furthermore, it is Hermitian with respect to the restriction  $(-, -)_{L^\nu}$  of  $(-, -)_{L^\omega}$ . That is,  $(L^\nu, (-, -)_{L^\nu}, \nabla^N)$  is a prequantisation of the action of  $K$  on  $N$ .

In the same way, we see that a  $\text{Spin}^c$ -prequantum line bundle on  $(M, \omega)$ , that is, a prequantum line bundle on  $(M, 2\omega)$ , restricts to a  $\text{Spin}^c$ -prequantum line bundle on  $(N, 2\nu)$ .

### Induction: an auxiliary connection $\nabla$

Now let us consider induction of prequantisations. As in Section 12.2, let  $(N, \nu)$  be a Hamiltonian  $K$ -manifold, with momentum map  $\Phi^N$ . Let  $(M, \omega, \Phi^M)$  be the Hamiltonian induction of these data. Let  $(L^\nu, (-, -)_{L^\nu}, \nabla^N)$  be an equivariant prequantisation of the action of  $K$  on  $N$ . As in the case of restriction, the following argument extends directly to  $\text{Spin}^c$ -prequantisations.

Consider the line bundle

$$L^\omega := G \times_K L^\nu \rightarrow M,$$

with the natural projection map  $[g, l] \mapsto [g, n]$  for  $g \in G, n \in N$  and  $l \in L_n^\nu$ . Let  $(-, -)_{L^\omega}$  be the  $G$ -invariant Hermitian metric on  $L^\omega$  induced by  $(-, -)_{L^\nu}$ : for all  $g, g' \in G, n \in N$  and  $l, l' \in L_n^\nu$ , set

$$([g, l], [g', l'])_{L^\omega} := (l, l')_{L^\nu}.$$

In the remainder of this section, we will construct a connection  $\nabla^M$  on  $L^\omega$ , such that  $(L^\omega, (-, -)_{L^\omega}, \nabla^M)$  is a  $G$ -equivariant prequantisation of  $(M, \omega)$ . This is by definition the prequantisation induced by  $(L^\nu, (-, -)_{L^\nu}, \nabla^N)$ .

To construct the connection  $\nabla^M$ , we consider the line bundle

$$L := G \times L^\nu \rightarrow G \times N,$$

with the obvious projection map  $(g, l) \mapsto (g, n)$ , for all  $g \in G, l \in L_n^\nu$ . Then  $L^\omega = L/K$ , where  $K$  acts on  $L$  by

$$k \cdot (g, l) = (gk^{-1}, k \cdot l),$$

for  $k \in K, g \in G$  and  $l \in L^\nu$ . By Proposition 8.6, we therefore have a linear isomorphism

$$\psi_L : \Gamma^\infty(L)^K \rightarrow \Gamma^\infty(L^\omega),$$

given by

$$\psi_L(\sigma)[g, n] = [\sigma(g, n)]. \quad (13.1)$$

We will construct  $\nabla^M$  as the connection induced by a  $K$ -equivariant connection  $\nabla$  on  $L$ . The space  $\Gamma^\infty(L)$  of sections of  $L$  is isomorphic to the space

$$\tilde{\Gamma}^\infty(L) := \{s : G \times N \xrightarrow{C^\infty} L^\nu; s(g, n) \in L_n^\nu \text{ for all } g \in G \text{ and } n \in N.\}$$

Indeed, the isomorphism is given by  $s \mapsto \sigma$ , where  $\sigma(g, n) = (g, s(g, n))$ . For  $s \in \tilde{\Gamma}^\infty(L)$ ,  $g \in G$  and  $n \in N$ , we write

$$s_g(n) := s(g, n) =: s^n(g).$$

(We will use the same notation when  $s$  is replaced by a function on  $G \times N$ .) Then for fixed  $g$ ,  $s_g$  is a section of  $L^\nu$ , and for fixed  $n$ ,  $s^n$  is a function

$$s^n : G \rightarrow L_n^\nu.$$

Let  $s \in \tilde{\Gamma}^\infty(L)$ ,  $X \in \mathfrak{g}$ ,  $\nu \in \mathfrak{X}(N)$ ,  $g \in G$  and  $n \in N$  be given. We define

$$(\nabla_{\nu+X}s)(g, n) := (\nabla_\nu^N s_g)(n) + X(s^n)(g) + 2\pi i \Phi_{X_t}^N(n) s(g, n). \quad (13.2)$$

Here we have written  $X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$ . (The subscript  $\mathfrak{k}$  in  $X_{\mathfrak{k}}$  in (13.2) is actually superfluous, because we identify  $\mathfrak{k}^*$  with  $\mathfrak{p}^0 \subset \mathfrak{g}^*$ .) In the expression  $X(s^n)$ , we view  $X$  as a left invariant vector field on  $G$ , acting on the function  $s^n$ . Note that all tangent vectors in  $T_{(g,n)}(G \times N)$  are of the form  $X_g + v_n = (g, X, v_n) \in T_g G \times T_n N$ , and therefore the above formula determines  $\nabla$  uniquely. We claim that  $\nabla$  is a  $K$ -equivariant connection on  $L$  with the right curvature, so that it induces a connection  $\nabla^M$  on  $L^\omega$  with curvature  $\omega$ .

**Lemma 13.1.** *The formula (13.2) defines a connection  $\nabla$  on  $L$ .*

*Proof.* The Leibniz rule for  $\nabla$  follows from the fact that for  $f \in C^\infty(G \times N)$ ,  $X \in \mathfrak{g}$ ,  $v \in \mathfrak{X}(N)$ ,  $g \in G$  and  $n \in N$ , one has

$$(v + X)(f)(g, n) = v(f_g)(n) + X(f^n)(g).$$

Linearity over  $C^\infty(G \times N)$  in the vector fields follows from the fact that, with notation as above,

$$(f(v + X))_{(g,n)} = (f^n X)_g + (f_g v)(n).$$

Locality is obvious. □

### Properties of the connection $\nabla$

Let  $(-, -)_L$  be the Hermitian metric on  $L$  given by

$$((g, l), (g', l'))_L := (l, l')_{L^V}$$

for all  $g, g' \in G$  and  $l, l' \in L_n^V$ .

**Lemma 13.2.** *The connection  $\nabla$  is Hermitian with respect to this metric.*

*Proof.* Let  $s, t \in \tilde{\Gamma}^\infty(L)$ ,  $X \in \mathfrak{g}$ ,  $v \in \mathfrak{X}(N)$ ,  $g \in G$  and  $n \in N$  be given. Then

$$\begin{aligned} (\nabla_{v+X}s, t)_L(g, n) + (s, \nabla_{v+X}t)_L(g, n) = \\ ((\nabla_v^N s_g)(n), t(g, n))_L + (s(g, n), (\nabla_v^N t_g)(n))_L \\ + (X(s^n)(g), t(g, n))_L + (s(g, n), X(t^n)(g))_L \\ + (2\pi i \Phi_{X_{\mathfrak{k}}}^N(n) s(g, n), t(g, n))_L + (s(g, n), 2\pi i \Phi_{X_{\mathfrak{k}}}^N(n) t(g, n))_L. \end{aligned}$$

By sesquilinearity of  $(-, -)_L$ , the last two terms cancel. And since  $\nabla^N$  is Hermitian, we are left with

$$v((s, t)_L)(g, n) + X((s, t)_L)(g, n) = (v + X)((s, t)_L)(g, n),$$

which shows that  $\nabla$  is indeed Hermitian. □

Next, we compute the curvature of  $\nabla$ .

**Lemma 13.3.** *The curvature  $R_\nabla$  of  $\nabla$  is given by*

$$R_\nabla(v + X, w + Y)(g, n) = 2\pi i (v_n(v, w) - \langle \Phi^N(n), [X, Y]_{\mathfrak{k}} \rangle),$$

for all  $X, Y \in \mathfrak{g}$ ,  $v, w \in \mathfrak{X}(N)$ ,  $g \in G$  and  $n \in N$ .

*Proof.* We compute:

$$\begin{aligned}
 (\nabla_{v+X} \nabla_{w+Y} s)(g, n) = & \\
 & (\nabla_v^N \nabla_w^N s_g)(n) + \nabla_v^N (n' \mapsto (Y s^{n'})(g))(n) \\
 & + X(g' \mapsto (\nabla_w^N s_{g'})(n))(g) + 2\pi i \Phi_{Y_\sharp}^N(n) (X s^n)(g) \\
 & + (X Y s^n)(g) + 2\pi i (v(\Phi_{Y_\sharp}^N s_g))(n) \\
 & + 2\pi i (\Phi_{Y_\sharp}^N \nabla_v^N s_g)(n) + 2\pi i (\Phi_{X_\sharp}^N \nabla_w^N s_g)(n) \\
 & - 4\pi^2 (\Phi_{X_\sharp}^N \Phi_{Y_\sharp}^N s_g)(n) + 2\pi i \Phi_{X_\sharp}^N(n) Y(s^n)(g).
 \end{aligned} \tag{13.3}$$

In this expression, the following terms are symmetric in  $v+X$  and  $w+Y$ :

- $2\pi i \Phi_{Y_\sharp}^N(n) (X s^n)(g) + 2\pi i \Phi_{X_\sharp}^N(n) Y(s^n)(g);$
- $2\pi i (\Phi_{Y_\sharp}^N \nabla_v^N s_g)(n) + 2\pi i (\Phi_{X_\sharp}^N \nabla_w^N s_g)(n);$
- $-4\pi^2 (\Phi_{X_\sharp}^N \Phi_{Y_\sharp}^N s_g)(n).$

Furthermore, note that

$$\begin{aligned}
 \nabla_v^N (n' \mapsto (Y s^{n'})(g))(n) &= \nabla_v^N (n' \mapsto \left. \frac{d}{dt} \right|_{t=0} s(\exp(-tY)g, n'))(n) \\
 &= \left. \frac{d}{dt} \right|_{t=0} (\nabla_v^N s_{\exp(-tY)g})(n) \\
 &= Y(g' \mapsto (\nabla_v^N s_{g'})(n))(g).
 \end{aligned}$$

Therefore, the following term in (13.3) is also symmetric in  $v+X$  and  $w+Y$ :

$$\nabla_v^N (n' \mapsto (Y s^{n'})(g))(n) + X(g' \mapsto (\nabla_w^N s_{g'})(n))(g).$$

We conclude that in the commutator  $[\nabla_{v+X}, \nabla_{w+Y}]$ , most terms in (13.3) drop out, and we are left with

$$([\nabla_{v+X}, \nabla_{w+Y}]s)(g, n) = ([\nabla_v^N, \nabla_w^N] s_g)(n) + ([X, Y] s^n)(g). \tag{13.4}$$

On the other hand, note that as vector fields on  $G \times N$ , the Lie brackets  $[X, v]$  and  $[Y, w]$  vanish. Therefore,

$$[v+X, w+Y] = [X, Y] + [v, w],$$

so that

$$\begin{aligned}
 (\nabla_{[v+X, w+Y]} s)(g, n) &= (\nabla_{[X, Y] + [v, w]} s)(g, n) \\
 &= (\nabla_{[v, w]}^N s_g)(n) + ([X, Y] s^n)(g) + 2\pi i \Phi_{[X, Y]_\sharp}^N(n) s(g, n)
 \end{aligned} \tag{13.5}$$

Finally, taking the difference of (13.4) and (13.5), we obtain

$$\begin{aligned}
 (R_{\nabla}(v+X, w+Y)s)(g, n) &= (R_{\nabla^N}(v, w)s_g)(n) - 2\pi i \Phi_{[X, Y]_\sharp}^N(n) s(g, n) \\
 &= 2\pi i (v_n(v_n, w_n) - \langle \Phi^N(n), [X, Y]_\sharp \rangle) s(g, n).
 \end{aligned}$$

□

It remains to show that the connection  $\nabla$  induces the desired connection  $\nabla^M$  on  $L^\omega$ . This will follow from  $K$ -equivariance of  $\nabla$ .

**Lemma 13.4.** *The connection  $\nabla$  is  $K$ -equivariant in the sense that for all  $X \in \mathfrak{g}$ ,  $v \in \mathfrak{X}(N)$ ,  $k \in K$ ,  $s \in \Gamma^\infty(L)$ ,  $g \in G$  and  $n \in n$ , we have*

$$k \cdot (\nabla_{v+X}s) = \nabla_{k \cdot (v+X)} k \cdot s.$$

*Proof.* By definition of the connection  $\nabla$ , we have

$$\begin{aligned} (k \cdot (\nabla_{v+X}s))(g, n) = \\ k \cdot ((\nabla_v^N s_{gk})(k^{-1}n)) + k \cdot (X(s^{k^{-1}n})(gk)) + \Phi_{X_\mathfrak{k}}^N(k^{-1}n)k \cdot (s(gk, k^{-1}n)). \end{aligned} \quad (13.6)$$

We examine this expression term by term.

By  $K$ -equivariance of  $\nabla^N$ , the first term in (13.6) equals

$$\begin{aligned} k \cdot ((\nabla_v^N s_{gk})(k^{-1}n)) &= (k \cdot (\nabla_v^N s_{gk}))(n) \\ &= (\nabla_{k \cdot v}^N k \cdot s_{gk})(n) \\ &= (\nabla_{k \cdot v}^N (k \cdot s)_g)(n). \end{aligned}$$

The second term equals

$$\begin{aligned} k \cdot (X(s^{k^{-1}n})(gk)) &= k \cdot \left. \frac{d}{dt} \right|_{t=0} s(gk \exp(tX), k^{-1}n) \\ &= k \cdot \left. \frac{d}{dt} \right|_{t=0} s(g \exp(t \operatorname{Ad}(k)X)k, k^{-1}n) \\ &= k \cdot (\operatorname{Ad}(k)X(s^{k^{-1}n})). \end{aligned}$$

Furthermore, note that for all  $g \in G$  and  $n \in N$ , we have

$$\begin{aligned} (\operatorname{Ad}(k)X)_{G \times N}(g, n) &= \left. \frac{d}{dt} \right|_{t=0} (\exp(t \operatorname{Ad}(k)X)g, n) \\ &= \left( \left. \frac{d}{dt} \right|_{t=0} k \exp(tX)k^{-1}g, 0 \right) \\ &= (T_{k^{-1}g}k(X_{k^{-1}g}), 0) \\ &= (k \cdot (X_{G \times N}))(g, n). \end{aligned}$$

Finally, by  $K$ -equivariance of  $\Phi^N$ , the last term in (13.6) is

$$\langle \Phi^N(k^{-1}n), X_\mathfrak{k} \rangle k \cdot (s(gk, k^{-1}n)) = \langle \Phi^N(n), \operatorname{Ad}(k)X_\mathfrak{k} \rangle (k \cdot s)(g, n).$$

Therefore,

$$(k \cdot (\nabla_{v+X}s))(g, n) = (\nabla_{k \cdot X + k \cdot v} k \cdot s)(g, n).$$

□

We now define  $\nabla^M$  via the isomorphism  $\psi_L$  in (13.1). Note that by Proposition 12.1 and Lemma 12.2, we have

$$\begin{aligned}\mathfrak{X}(M) &\cong \Gamma^\infty(G \times_K N, G \times_K (TN \times \mathfrak{p})) \\ &\cong \Gamma^\infty(G \times N, G \times TN \times \mathfrak{p})^K \\ &\subset \Gamma^\infty(G \times N, (G \times \mathfrak{g}) \times TN)^K \\ &= \mathfrak{X}(G \times N)^K\end{aligned}$$

We will write  $j : \mathfrak{X}(M) \hookrightarrow \mathfrak{X}(G \times N)^K$  for this embedding map. For  $w \in \mathfrak{X}(M)$  and  $s \in \Gamma^\infty(L)^K$ , we define the connection  $\nabla^M$  by

$$\nabla_w^M \psi_L(s) := \psi_L(\nabla_{j(w)} s).$$

Because  $s$  and  $j(w)$  are  $K$ -invariant, and  $\nabla$  is  $K$ -equivariant, we indeed have  $\nabla_{j(w)} s \in \Gamma^\infty(L)^K$ , the domain of  $\psi_L$ .

It now follows directly from the definitions and from Lemmas 13.1, 13.2 and 13.3 that  $\nabla^M$  is a Hermitian connection on  $L^\omega$  with curvature  $\omega$ .

## Induction and restriction

The induction and restriction procedures for line bundles described above are each other's inverses (modulo equivariant line bundle isomorphisms), although this does not include the connections on the bundles in question:

**Lemma 13.5.** (i) *Let  $N$  be a  $K$ -manifold, and  $q^N : E^N \rightarrow N$  a  $K$ -vector bundle. Then*

$$(G \times_K E^N)|_{\tilde{N}} \cong E^N,$$

*with  $\tilde{N}$  as in (12.9).*

(ii) *Let  $M$  be a  $G$ -manifold,  $E^M \rightarrow M$  a  $G$ -vector bundle. Let  $N \subset M$  be a  $K$ -invariant submanifold, and denote the restriction of  $E^M$  to  $N$  by  $E^N$ . Let  $\phi : G \times_K N \rightarrow M$  be the map  $\phi[g, n] = gn$ . Then*

$$\phi^* E^M \cong G \times_K E^N.$$

*Proof.* (i) Note that

$$\begin{aligned}(G \times_K E^N)|_{\tilde{N}} &= \{[g, v] \in G \times_K E^N; [g, q^N(v)] = [e, n] \text{ for an } n \in N\} \\ &= \{[e, v] \in G \times_K E^N; v \in E^N\} \\ &\cong E^N.\end{aligned}$$

(ii) Note that

$$\phi^* E^M = \{([g, n], v); g \in G, n \in N \text{ and } v \in E_{gn}^M\}.$$

The map  $([g, n], v) \mapsto [g, v]$  is the desired vector bundle isomorphism onto  $G \times_K E^N$ .  $\square$

For our purposes, it does not matter that this lemma says nothing about connections that may be defined on the vector bundles in question, because the  $K$ -homology classes defined by Dirac operators associated to such connections are homotopy invariant. In our setting, the vector bundle isomorphisms in the proof of Lemma 13.5 do intertwine the metrics  $(-, -)_{L^\omega}$  and  $(-, -)_{L^v}$  on the respective line bundles.

## 13.2 $\text{Spin}^c$ -structures

Because we want to compare the Dirac operators on  $M$  and  $N$ , we now look at induction of  $\text{Spin}^c$ -structures. As before, we consider a semisimple group  $G$  with maximal compact subgroup  $K$ , and a  $K$ -manifold  $N$ . We form the fibred product  $M := G \times_K N$ , and we will show how a  $K$ -equivariant  $\text{Spin}^c$ -structure on  $N$  induces a  $G$ -equivariant  $\text{Spin}^c$ -structure on  $M$ . It will turn out that the operation of taking determinant line bundles intertwines the induction process for  $\text{Spin}^c$ -structures in this section, and the induction process for prequantum line bundles in the previous one.

### General constructions

The construction of induced  $\text{Spin}^c$ -structures we will use, is based on the following two facts, of which we were informed by Paul-Émile Paradan.

**Lemma 13.6.** *For  $j = 1, 2$ , let  $E_j \rightarrow M$  be a real vector bundle over a manifold  $M$ . Suppose  $E_1$  and  $E_2$  are equipped with metrics and orientations. Let  $P_j \rightarrow M$  be a  $\text{Spin}^c$ -structure on  $E_j$ , with determinant line bundle  $L_j \rightarrow M$ . Then there is a  $\text{Spin}^c$ -structure  $P \rightarrow M$  on the direct sum  $E_1 \oplus E_2 \rightarrow M$ , with determinant line bundle  $L_1 \otimes L_2$ .*

*Proof.* Let  $r_j$  be the rank of  $E_j$ , and write  $r := r_1 + r_2$ . Consider the double covering map

$$\pi : \text{Spin}^c(r) \rightarrow \text{SO}(r) \times \text{U}(1),$$

given by  $[a, z] \mapsto (\lambda(a), z)$ , where  $a \in \text{Spin}(r)$ ,  $z \in \text{U}(1)$ , and  $\lambda : \text{Spin}(r) \rightarrow \text{SO}(r)$  is the standard double covering. Consider the subgroups

$$H' := \text{SO}(r_1) \times \text{SO}(r_2) \times \text{U}(1)$$

of  $\text{SO}(r) \times \text{U}(1)$ , and  $H := \pi^{-1}(H')$  of  $\text{Spin}^c(r)$ . Noting that

$$H' \cong (\text{SO}(r_1) \times \text{U}(1)) \times_{\text{U}(1)} (\text{SO}(r_2) \times \text{U}(1)),$$

we see that

$$H \cong \text{Spin}^c(r_1) \times_{\text{U}(1)} \text{Spin}^c(r_2).$$

Let  $P_1 \times_{\text{U}(1)} P_2$  be the quotient of  $P_1 \times P_2$  by the  $\text{U}(1)$ -action given by

$$z(p_1, p_2) = (p_1 z, p_2 z^{-1}),$$

for  $z \in \text{U}(1)$  and  $p_j \in P_j$ . Define

$$P := (P_1 \times_{\text{U}(1)} P_2) \times_H \text{Spin}^c(r).$$

Then we have naturally defined isomorphisms

$$\begin{aligned} P \times_{\text{Spin}^c(r)} \mathbb{R}^r &\cong (P_1 \times_{\text{U}(1)} P_2) \times_H (\mathbb{R}^{r_1} \oplus \mathbb{R}^{r_2}) \\ &\cong (P_1 \times_{\text{Spin}^c(r_1)} \mathbb{R}^{r_1}) \oplus (P_2 \times_{\text{Spin}^c(r_2)} \mathbb{R}^{r_2}) \\ &\cong E_1 \oplus E_2. \end{aligned}$$

The determinant line bundle of  $P$  is

$$\det(P) = (P_1 \times_{U(1)} P_2) \times_H \mathbb{C},$$

where  $H$  acts on  $\mathbb{C}$  via the determinant homomorphism. Note that, for all  $h = [h_1, h_2] \in \text{Spin}^c(r_1) \times_{U(1)} \text{Spin}^c(r_2) \cong H$ , we have  $\det(h) = \det(h_1) \det(h_2)$ . Using this equality, one can check that the map

$$(P_1 \times_{U(1)} P_2) \times_H \mathbb{C} \rightarrow (P_1 \times_{\text{Spin}^c(r_1)} \mathbb{C}) \otimes (P_2 \times_{\text{Spin}^c(r_2)} \mathbb{C}),$$

given by

$$[p_1, p_2, z] \mapsto [p_1, z] \otimes [p_2, 1],$$

defines an isomorphism  $\det(P) \cong \det(P_1) \otimes \det(P_2)$ .  $\square$

**Lemma 13.7.** *Let  $G$  be a Lie group, acting on a smooth manifold  $N$ . Let  $H < G$  be a closed subgroup, and consider the fibred product  $M := G \times_H N$ . Let  $E^N \rightarrow N$  be an oriented  $H$ -vector bundle of rank  $r$ , equipped with an  $H$ -invariant metric. Then, as in Section 13.1, we can form the  $G$ -vector bundle*

$$E^M := G \times_H E^N \rightarrow M.$$

*If  $P^N \rightarrow N$  is an  $H$ -equivariant  $\text{Spin}^c$ -structure on  $E$ , then  $P^M := G \times_H P^N$  is a  $G$ -invariant  $\text{Spin}^c$ -structure on  $E^M$ . If  $L^N \rightarrow N$  is the determinant line bundle of  $P^N$ , then the determinant line bundle of  $P^M$  is  $G \times_H L^N$ .*

*Proof.* The first claim is a direct consequence of the fact that the actions of  $H$  and  $\text{Spin}^c(r)$  on  $P^N$  commute. For the same reason, we have

$$\begin{aligned} \det(P^M) &= (G \times_H P^N) \times_{\text{Spin}^c(r)} \mathbb{C} \\ &= G \times_H (P^N \times_{\text{Spin}^c(r)} \mathbb{C}) \\ &= G \times_H L^N. \end{aligned}$$

$\square$

## An induced $\text{Spin}^c$ -structure

Let a  $K$ -equivariant  $\text{Spin}^c$ -structure  $P^N$  on  $N$  be given. To construct a  $G$ -equivariant  $\text{Spin}^c$ -structure on  $M = G \times_K N$ , we recall that, by Corollary 12.3,

$$TM \cong (p_{G/K}^* T(G/K)) \oplus (G \times_K TN), \quad (13.7)$$

with  $p_{G/K} : M \rightarrow G/K$  the natural projection. As in Section 6.2, we assume that the homomorphism  $\text{Ad} : K \rightarrow \text{SO}(\mathfrak{p})$  lifts to a homomorphism  $\widetilde{\text{Ad}} : K \rightarrow \text{Spin}(\mathfrak{p})$ . Then  $G/K$  carries the natural  $\text{Spin}$ -structure

$$P^{G/K} := G \times_K \text{Spin}(\mathfrak{p}),$$

where  $K$  acts on  $\text{Spin}(\mathfrak{p})$  via  $\widetilde{\text{Ad}}$ .



**Lemma 13.8.** *The principal  $\text{Spin}^c(\mathfrak{p})$ -bundle*

$$P_M^{G/K} := G \times_K (N \times \text{Spin}^c(\mathfrak{p})) \rightarrow M$$

*defines a  $\text{Spin}^c$ -structure on  $p_{G/K}^* T(G/K)$ . Its determinant line bundle is trivial.*

*Proof.* We have

$$\begin{aligned} G \times_K (N \times \text{Spin}^c(\mathfrak{p})) \times_{\text{Spin}^c(\mathfrak{p})} \mathfrak{p} &\cong G \times_K (N \times \mathfrak{p}) \\ &\cong p_{G/K}^* (G \times_K \mathfrak{p}) \\ &\cong p_{G/K}^* T(G/K). \end{aligned}$$

Note that the determinant homomorphism is trivial on the subgroup  $\text{Spin}(\mathfrak{p}) < \text{Spin}^c(\mathfrak{p})$ , and that  $\widetilde{\text{Ad}}(K) < \text{Spin}(\mathfrak{p})$ . Therefore, the action of  $K$  on  $\mathbb{C}$ , given by the composition

$$K \xrightarrow{\widetilde{\text{Ad}}} \text{Spin}(\mathfrak{p}) \hookrightarrow \text{Spin}^c(\mathfrak{p}) \xrightarrow{\det} \text{U}(1),$$

is trivial. We conclude that

$$\det(P_M^{G/K}) \cong G \times_K (N \times \mathbb{C}) \cong M \times \mathbb{C},$$

as claimed. □

Using the decomposition (13.7) of  $TM$ , and the constructions from Lemmas 13.6 and 13.7, we now obtain a  $\text{Spin}^c$ -structure  $P^M \rightarrow M$  on  $M$ , from the  $\text{Spin}^c$ -structures  $P_M^{G/K} \rightarrow M$  and  $P^N \rightarrow N$ . Explicitly,

$$P^M := (G \times_K (N \times \text{Spin}^c(\mathfrak{p}))) \times_{\text{U}(1)} (G \times_K P^N) \times_H \text{Spin}^c(d_M).$$

By Lemmas 13.6 and 13.7, and by triviality of  $\det(P_M^{G/K})$ , we see that the determinant line bundle of  $P^M$  equals

$$\det(P^M) = G \times_K \det(P^N).$$

In particular, if the determinant line bundle of  $P^N$  is a  $\text{Spin}^c$ -prequantum line bundle  $L^{2\nu} \rightarrow N$ , then

$$\det(P^M) = G \times_K L^{2\nu} = L^\omega \tag{13.8}$$

is the  $\text{Spin}^c$ -prequantum line bundle on  $M$  constructed in Section 13.1.

# Chapter 14

## Quantisation commutes with induction

Our proof that quantisation commutes with reduction for semisimple groups is a reduction to the case of compact groups. This reduction is possible because of the ‘quantisation commutes with induction’ result in this chapter (Theorem 14.5). It is analogous to Theorem 7.5 from [63]. After stating this result, we show how, together with the quantisation commutes with reduction result for the compact case, it implies Theorem 6.13. Our proof that quantisation commutes with induction is based on naturality of the assembly map for the inclusion  $K \hookrightarrow G$  (Theorem 9.1). This proof is outlined in Section 14.4, with details given in Chapter 15.

### 14.1 The sets $\text{CSEHamPS}(G)$ and $\text{CSEHamPS}(K)$

We first restate the results of Chapters 12 and 13 in a way that will allow us to draw a ‘quantisation commutes with induction’ diagram.

**Definition 14.1.** The set  $\text{SEHamP}(G)$  of *Hamiltonian*  $G$ -actions with momentum map values in the strongly elliptic set, with  $\text{Spin}^c$ -prequantisations, consists of classes of sextuples  $(M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M)$ , where

- $(M, \omega)$  is a symplectic manifold, equipped with a symplectic  $G$ -action;
- $\Phi^M : M \rightarrow \mathfrak{g}^*$  is a momentum map for this action, and  $\Phi^M(M) \subset \mathfrak{g}_{\text{se}}^*$ ;
- $(L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M)$  is a  $G$ -equivariant  $\text{Spin}^c$ -quantisation of  $(M, \omega)$ .

Two classes  $[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M]$  and  $[M', \omega', \Phi^{M'}, L^{2\omega'}, (-, -)_{L^{2\omega'}}, \nabla^{M'}]$  of such sextuples are identified if there is an equivariant symplectomorphism  $\varphi : M \rightarrow M'$  such that  $\varphi^* \Phi^{M'} = \Phi^M$ ,  $\varphi^* L^{2\omega'} = L^{2\omega}$  and  $\varphi^* (-, -)_{L^{2\omega'}} = (-, -)_{L^{2\omega}}$ . We do not require  $\varphi$  to relate the connections  $\nabla^M$  and  $\nabla^{M'}$  to each other. For the purpose of quantisation, it is enough that it relates their curvatures by  $\varphi^* R_{\nabla^{M'}} = R_{\nabla^M}$ , which follows from the facts that  $\varphi$  is a symplectomorphism, and that  $\nabla^M$  and  $\nabla^{M'}$  are prequantum connections.

Analogously,  $\text{SEHamP}(K)$  is the set of classes  $[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N]$ , where  $(N, \nu)$  is a Hamiltonian  $K$ -manifold, with momentum map  $\Phi^N$ , with image in  $\mathfrak{k}_{\text{se}}^*$ , and  $(L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N)$  is a  $K$ -equivariant  $\text{Spin}^c$ -prequantisation of  $(N, \nu)$ . The equivalence relation between these classes is the same as before.

Using this definition, we can summarise the results of Sections 12.2, 12.3, 12.4 and 13.1 as follows:

**Theorem 14.2.** *There are well-defined maps*

$$\text{H-Ind}_K^G : \text{SEHamP}(K) \rightarrow \text{SEHamP}(G)$$

and

$$\text{H-Cross}_K^G : \text{SEHamP}(G) \rightarrow \text{SEHamP}(K),$$

given by

$$\text{H-Ind}_K^G[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N] = [M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M]$$

as in Sections 12.2 and 13.1, and

$$\text{H-Cross}_K^G[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M] = [N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N]$$

as in Sections 12.3 and 13.1. They are each other's inverses.

To state our ‘quantisation commutes with reduction’ result, we need slightly different sets from  $\text{SEHamP}(G)$  and  $\text{SEHamP}(K)$ . For these sets we only have an induction map, and we do not know if it is possible to define a suitable cross-section map.

**Definition 14.3.** The set  $\text{CSEHamPS}(G)$  of *cocompact Hamiltonian  $G$ -actions on complete manifolds, with momentum map values in the strongly elliptic set, with  $\text{Spin}^c$ -prequantisations and  $\text{Spin}^c$ -structures*, consists of classes of septuples  $(M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M)$ , with  $(M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M)$  as in Definition 14.1,  $M/G$  compact, and  $P^M$  a  $G$ -equivariant  $\text{Spin}^c$ -structure on  $M$ , such that

- $M$  is complete in the Riemannian metric induced by  $P^M$ ;
- the determinant line bundle of  $P^M$  is isomorphic to  $L^{2\omega}$ .

The equivalence relation is the same as in Definition 14.1. There is no need to incorporate the  $\text{Spin}^c$ -structures into this equivalence relation, besides the condition on the determinant line bundles of these structures that is already present.

The set  $\text{CSEHamPS}(K)$  is defined analogously. In this case, the condition that  $N/K$  is compact is equivalent to compactness of  $N$ .

For these sets, we have the induction map

$$\text{H-Ind}_K^G : \text{CSEHamPS}(K) \rightarrow \text{CSEHamPS}(G), \quad (14.1)$$

with

$$\text{H-Ind}_K^G[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] = [M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M],$$

as defined in Sections 12.2, 13.1 and 13.2.

## 14.2 Quantisation commutes with induction

Consider an element  $[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M] \in \text{CSEHamPS}(G)$ . Using a connection on the spinor bundle associated to  $P^M$ , we can define the  $\text{Spin}^c$ -Dirac operator  $\not{D}_M^{L^{2\omega}}$  on  $M$ , as in Section 3.4. In Definition 6.2, we defined the quantisation of the action of  $G$  on  $(M, \omega)$  as the image of the  $K$ -homology class of  $\not{D}_M^{L^{2\omega}}$  under the analytic assembly map:

$$Q_{VI}(M, \omega) = \mu_M^G[\not{D}_M^{L^{2\omega}}].$$

as we noted before, this definition does not depend on the choice of connection on the spinor bundle.

**Definition 14.4.** The quantisation map

$$Q_{VI}^G : \text{CSEHamPS}(G) \rightarrow K_0(C_r^*(G))$$

is defined by

$$Q_{VI}^G[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M] = \mu_M^G[\not{D}_M^{L^{2\omega}}].$$

Analogously, we have the quantisation map

$$Q_{VI}^K : \text{CSEHamPS}(K) \rightarrow K_0(C_r^*(K))$$

given by

$$Q_{VI}^K[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] = \mu_N^K[\not{D}_N^{L^{2\nu}}],$$

which corresponds to  $K$ -index  $\not{D}_N^{L^{2\nu}} \in R(K)$  by Proposition 5.17.

Using the Dirac induction map (6.10) and the Hamiltonian induction map (14.1), we can now state the following result:

**Theorem 14.5** (Quantisation commutes with induction). *The following diagram commutes:*

$$\begin{array}{ccc} \text{CSEHamPS}(G) & \xrightarrow{Q_{VI}^G} & K_0(C_r^*(G)) \\ \text{H-Ind}_K^G \uparrow & & \uparrow \text{D-Ind}_K^G \\ \text{CSEHamPS}(K) & \xrightarrow{Q_{VI}^K} & R(K). \end{array} \quad (14.2)$$

This is the central result of Part IV. We will outline its proof in Section 14.4, and fill in the details in Chapter 15.

## 14.3 Corollary: $[Q, R] = 0$ for semisimple groups

As announced, we derive Theorem 6.13 from Theorem 14.5 and the fact that  $\text{Spin}^c$ -quantisation commutes with reduction in the compact case (Theorem 3.38).

*Proof of Theorem 6.13.* Let  $G, K, (M, \omega), \Phi^M = \Phi, L^{2\omega} = L, (-, -)_{L^{2\omega}} = (-, -)_L$  and  $\nabla^M = \nabla$  be as in Theorem 6.13. Set

$$(N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N) := \text{H-Cross}_K^G(M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M).$$

Let  $P^N \rightarrow N$  be a  $K$ -equivariant  $\text{Spin}^c$ -structure on  $N$ , with determinant line bundle  $L^{2\nu}$ . Let  $P^M \rightarrow M$  be the induced  $\text{Spin}^c$ -structure on  $M$ , as described in Section 13.2. Since the determinant line bundle of  $P^M$  is  $L^{2\omega}$ , by (13.8) and part (ii) of Lemma 13.5, we have the elements

$$\begin{aligned} [N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] &\in \text{CSEHamPS}(K); \\ [M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M] &\in \text{CSEHamPS}(G). \end{aligned}$$

By Proposition 12.15, we have

$$\text{H-Ind}_K^G[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] = [M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M].$$

Now let  $\mathcal{H}$  and  $\lambda$  be as in Theorem 6.13. Then by Theorem 14.5, Proposition 5.17 and Lemma 6.9, we have

$$\begin{aligned} R_G^{\mathcal{H}} \circ \mu_M^G[\mathcal{D}_M^{L^{2\omega}}] &= R_G^{\mathcal{H}} \circ \text{D-Ind}_K^G(K\text{-index } \mathcal{D}_N^{L^{2\nu}}) \\ &= (-1)^{\dim G/K} [K\text{-index } \mathcal{D}_N^{L^{2\nu}} : V_{\lambda - \rho_c}]. \end{aligned}$$

Because  $\text{Spin}^c$ -quantisation commutes with reduction for the action of  $K$  on  $N$  (Theorem 3.38), we have

$$[K\text{-index } \mathcal{D}_N^{L^{2\nu}} : V_{\lambda - \rho_c}] = \mathcal{Q}_N(N_\lambda, \omega_\lambda)$$

if  $-i\lambda \in \Phi^N(N)$ , and zero otherwise. Recall that  $N = (\Phi^M)^{-1}(\mathfrak{k}^*)$ , so that  $-i\lambda \in \Phi^N(N)$  if and only if  $-i\lambda \in \Phi^M(M)$ . Furthermore, note that  $G_\nu \subset K$  for all  $\nu \in \mathfrak{k}_+^* \setminus \text{ncw}$ , so that  $G_\nu = K_\nu$  for such  $\nu$ . Therefore  $N_\lambda = M_\lambda$ , which completes the proof.  $\square$

## 14.4 Outline of the proof

The most important ingredient of the proof of Theorem 14.5 is Theorem 9.1, ‘naturalness of the assembly map for the inclusion of  $K$  into  $G$ ’. The reason why this theorem helps us to prove Theorem 14.5 is the fact that the map  $\text{K-Ind}_K^G$  that appears in Theorem 9.1 relates the Dirac operators  $\mathcal{D}_N^{L^{2\nu}}$  and  $\mathcal{D}_M^{L^{2\omega}}$  to each other:

**Proposition 14.6.** *The map  $\text{K-Ind}_K^G$  maps the  $K$ -homology class of the operator  $\mathcal{D}_N^{L^{2\nu}}$  to the class of  $\mathcal{D}_M^{L^{2\omega}}$ .*

Combining Theorem 9.1 and Proposition 14.6, we obtain a proof of Theorem 14.5:

*Proof of Theorem 14.5.* Let

$$x = [N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] \in \text{CSEHamPS}(K)$$

be given, and write

$$[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M] := \text{H-Ind}_K^G(x).$$

Then by Proposition 14.6 and Theorem 9.1,

$$\begin{aligned} Q_{VI}^G(\text{H-Ind}_K^G(x)) &= \mu_M^G[\mathcal{D}_M^{L^{2\omega}}] \\ &= \mu_M^G \circ \text{K-Ind}_K^G[\mathcal{D}_N^{L^{2\nu}}] \\ &= \text{D-Ind}_K^G \circ \mu_N^K[\mathcal{D}_N^{L^{2\nu}}] \\ &= \text{D-Ind}_K^G(Q_{VI}^K(x)). \end{aligned}$$

□

It remains to prove Proposition 14.6. This proof will be given in Chapter 15.

# Chapter 15

## Dirac operators and the map $\text{K-Ind}_K^G$

This chapter is devoted to the proof of Proposition 14.6. We will define an operator  $\tilde{D}_M^{L^{2\omega}}$  whose  $K$ -homology class is the image of the class of  $\tilde{D}_N^{L^{2\nu}}$  under the map  $\text{K-Ind}_K^G$ . Then we prove some general facts about principal symbols, and finally we use these facts to show that  $\tilde{D}_M^{L^{2\omega}}$  and  $\tilde{D}_M^{L^{2\omega}}$  define the same class in  $K$ -homology, proving Proposition 14.6.

Throughout this chapter, we will consider a class

$$[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] \in \text{CSEHamPS}(K),$$

and we will write

$$[M, \omega, \Phi^M, L^{2\omega}, (-, -)_{L^{2\omega}}, \nabla^M, P^M] := \text{H-Ind}_K^G[N, \nu, \Phi^N, L^{2\nu}, (-, -)_{L^{2\nu}}, \nabla^N, P^N] \in \text{CSEHamPS}(G).$$

### 15.1 Another Dirac operator on $M$

Let us construct the differential operator  $\tilde{D}_M^{L^{2\omega}}$  mentioned in the introduction to this chapter. Just like the  $\text{Spin}^c$ -Dirac operator  $\tilde{D}_M^{L^{2\omega}}$ , it acts on sections of the spinor bundle

$$\mathcal{S}^M := P^M \times_{\text{Spin}^c(d_M)} \Delta_{d_M} \rightarrow M, \quad (15.1)$$

associated to the  $\text{Spin}^c$ -structure  $P^M$  defined in Section 13.2.

In the definition of the operator  $\tilde{D}_M^{L^{2\omega}}$ , we will use the following decomposition of the spinor bundle  $\mathcal{S}^M$ :

**Lemma 15.1.** *We have a  $G$ -equivariant isomorphism of vector bundles over  $M$ ,*

$$\mathcal{S}^M \cong ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N) / K,$$

where  $K$  acts on  $(G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N$  by

$$k \cdot ((g, \delta_p) \otimes s^N) = (gk^{-1}, \widetilde{\text{Ad}}(k)\delta_p) \otimes k \cdot s^N,$$

for  $k \in K$ ,  $g \in G$ ,  $\delta_p \in \Delta_{d_p}$  and  $s^N \in \mathcal{S}^N$ .

*Proof.* We have the following chain of isomorphisms:

$$\begin{aligned}
 \mathcal{S}^M &\cong (P_M^{G/K} \times_{U(1)} (G \times_K P^N)) \times_H \Delta_{d_p} \otimes \Delta_{d_N} \\
 &\cong (P_M^{G/K} \times_{\text{Spin}^c(d_p)} \Delta_{d_p}) \otimes (G \times_K P^N \times_{\text{Spin}^c(d_N)} \Delta_{d_N}) \\
 &\cong (G \times N \times \Delta_{d_p}) / K \otimes (G \times \mathcal{S}^N) / K \\
 &\cong ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N) / K.
 \end{aligned} \tag{15.2}$$

The first isomorphism in (15.2) is induced by the  $H$ -equivariant isomorphism  $\Delta_{d_M} \cong \Delta_{d_p} \otimes \Delta_{d_N}$ .

The second isomorphism is given by

$$[p_M^{G/K}, [g, p^N], \delta_p \otimes \delta_N] \mapsto [p_M^{G/K}, \delta_p] \otimes [[g, p^N], \delta_N],$$

for all  $p_M^{G/K} \in P_M^{G/K}$ ,  $g \in G$ ,  $p^N \in P^N$ ,  $\delta_p \in \Delta_{d_p}$  and  $\delta_N \in \Delta_{d_N}$ .

The third isomorphism is the obvious one, given the definitions of  $P_M^{G/K}$  and  $\mathcal{S}^N$ .

Finally, the fourth isomorphism is a special case of the isomorphism

$$E/G \otimes F/G \cong (E \otimes F)/G,$$

if  $H$  is a group acting freely on a manifold  $M$ , and  $E \rightarrow M$  and  $F \rightarrow M$  are  $G$ -vector bundles.

Explicitly, the isomorphism (15.2) is given by

$$[[g, n, a], [g, p^N], \delta_{d_p} \otimes \delta_N] \mapsto [(g, a\delta_p) \otimes [p^N, \delta_N]],$$

for  $g \in G$ ,  $n \in N$ ,  $a \in \text{Spin}^c(\mathfrak{p})$ ,  $p^N \in P^N$ ,  $\delta_p \in \Delta_{d_p}$  and  $\delta_N \in \Delta_{d_N}$ . □

Next, let  $\mathcal{D}_{G,K}$  be the operator defined on page 122, and consider the operator

$$\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^{L^{2v}} : \Gamma^\infty(G \times N, (G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N) \rightarrow \Gamma^\infty(G \times N, (G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N),$$

which is odd with respect to the grading on the tensor product  $(G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N$  induced by the gradings on  $\Delta_{d_p}$  and  $\mathcal{S}^N$ . Because the operators  $\mathcal{D}_{G,K}$  and  $\mathcal{D}_N^{L^{2v}}$  are  $K$ -equivariant, we obtain an operator

$$\tilde{\mathcal{D}}_M^{L^{2\omega}} := (\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^L)^K \tag{15.3}$$

on

$$\begin{aligned}
 \Gamma^\infty(G \times N, (G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)^K &\cong \Gamma^\infty(M, ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N \otimes) / K) \\
 &\cong \Gamma^\infty(M, \mathcal{S}^M),
 \end{aligned}$$

by Proposition 8.6 and Lemma 15.1.

The importance of the operator  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$  lies in the following fact:

**Lemma 15.2.** *The image of the class  $[\mathcal{D}_N^{L^{2v}}] \in K_0^K(N)$  under the map  $\text{K-Ind}_K^G$  is the class of  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$  in  $K_0^G(M)$ .*



*Proof.* By Theorem 10.8.7 from [34],<sup>1</sup> the Kasparov product  $[\mathcal{D}_{G,K}] \times [\mathcal{D}_N^{L^{2\nu}}] \in K_0^{G \times K \times K}(G \times N)$  is the class of the operator  $\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^{L^{2\nu}}$  on  $(G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N$ . It then follows from Corollary 8.11 that the latter class is mapped to the class of  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$ .  $\square$

Therefore, Proposition 14.6 follows if we can prove that  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$  and  $\mathcal{D}_M^{L^{2\omega}}$  define the same  $K$ -homology class. We prove this fact by showing that their principal symbols are equal (see Remark 4.34).

## 15.2 Principal symbols

This section contains some general facts about the principal symbols of differential operators that are constructed from other differential operators. These facts may be well-known and straightforward to prove, but we have included them here for completeness' sake.

### Tensor products

First, let  $X$  and  $Y$  be smooth manifolds, and let  $E \rightarrow X$  and  $F \rightarrow Y$  be vector bundles. Let  $D_E : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E)$  and  $D_F : \Gamma^\infty(F) \rightarrow \Gamma^\infty(F)$  be differential operators of the same order  $d$ . Consider the exterior tensor product  $E \boxtimes F \rightarrow X \times Y$ , and let  $D := D_E \otimes 1 + 1 \otimes D_F$  be the operator on  $\Gamma^\infty(E \boxtimes F)$  given by

$$D(s \boxtimes t) = D_E s \boxtimes t + s \boxtimes D_F t,$$

for  $s \in \Gamma^\infty(E)$  and  $t \in \Gamma^\infty(F)$ .

As before, we denote the cotangent bundle projection of a manifold  $M$  by  $\pi_M$ . The principal symbols of the operators  $D_E$ ,  $D_F$  and  $D$  are vector bundle homomorphisms

$$\begin{aligned} \sigma_{D_E} : \pi_X^* E &\rightarrow \pi_X^* E; \\ \sigma_{D_F} : \pi_Y^* F &\rightarrow \pi_Y^* F; \\ \sigma_D : \pi_{X \times Y}^* (E \boxtimes F) &\rightarrow \pi_{X \times Y}^* (E \boxtimes F). \end{aligned}$$

Let

$$\theta : \pi_{X \times Y}^* (E \boxtimes F) \rightarrow \pi_X^* E \boxtimes \pi_Y^* F$$

be the isomorphism of vector bundles over  $T^*(X \times Y) \cong T^*X \times T^*Y$  given by

$$\theta((\xi, \eta), (e \otimes f)) = (\xi, e) \otimes (\eta, f),$$

for  $x \in X$ ,  $y \in Y$ ,  $\xi \in T_x^* X$ ,  $\eta \in T_y^* Y$ ,  $e \in E_x$  and  $f \in F_y$ . The first fact about principal symbols that we will use is:

---

<sup>1</sup>This can also be seen in the unbounded picture of  $KK$ -theory.

**Lemma 15.3.** *The following diagram commutes:*

$$\begin{array}{ccc} \pi_{X \times Y}^*(E \boxtimes F) & \xrightarrow{\sigma_D} & \pi_{X \times Y}^*(E \boxtimes F) \\ \theta \downarrow \cong & & \theta \downarrow \cong \\ \pi_X^*E \boxtimes \pi_Y^*F & \xrightarrow{\sigma_{D_E} \otimes 1 + 1 \otimes \sigma_{D_F}} & \pi_X^*E \boxtimes \pi_Y^*F. \end{array}$$

*Proof.* Let  $g \in C^\infty(X)$ ,  $h \in C^\infty(Y)$ ,  $s \in \Gamma^\infty(E)$  and  $t \in \Gamma^\infty(F)$  be given. Let  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  be the natural projections. Then we have the function  $p_X^*g + p_Y^*h \in C^\infty(X \times Y)$ . Let  $x \in X$  and  $y \in Y$  be given. Set  $\mu := d_{(x,y)}(p_X^*g + p_Y^*h) \in T_{(x,y)}^*X \times Y$ . Note that all elements of this cotangent space can be written in this way (for certain functions  $g$  and  $h$ ). We compute:

$$\begin{aligned} \sigma_D(\mu, s(x) \otimes t(y)) &= \\ (\mu, \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^d} e^{-i\lambda(p_X^*g + p_Y^*h)} D(e^{i\lambda(p_X^*g + p_Y^*h)} s \boxtimes t)(x, y)) &= \\ (\mu, \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^d} [(e^{-i\lambda g} \otimes e^{-i\lambda h})(D_E(e^{i\lambda g} s) \boxtimes e^{i\lambda h} t + e^{i\lambda g} s \boxtimes D_F(e^{i\lambda h} t))](x, y)) &= \\ (\mu, \sigma_{D_E}(d_x g, s(x)) \otimes t(y) + s(x) \otimes \sigma_{D_F}(d_y h, t(y))) &. \end{aligned}$$

In other words,

$$\theta \circ \sigma_D \circ \theta^{-1}((d_x g, s(x)) \otimes (d_y h, t(y))) = (\sigma_{D_E} \otimes 1 + 1 \otimes \sigma_{D_F})((d_x g, s(x)) \otimes (d_y h, t(y))).$$

□

## Pullbacks

Next, let  $X$  and  $Y$  again be smooth manifolds, and let  $q : E \rightarrow Y$  be a vector bundle. Let  $f : X \rightarrow Y$  be a smooth map. (We will later apply this to the situation  $X = G \times N$ ,  $Y = M$ ,  $E = \mathcal{S}^M \otimes L^{2\omega}$ , and  $f$  the quotient map.) Let  $D_E$  be a differential operator on  $E$ , of order  $d$ . Let  $D_{f^*E}$  be a differential operator on the pullback bundle  $f^*E$  with the property that for all  $s \in \Gamma^\infty(E)$ ,

$$D_{f^*E}(f^*s) = f^*(D_E s).$$

Consider the vector bundle

$$f^*(T^*Y \oplus E) \rightarrow X.$$

It consists of triples  $(x, \xi, e) \in X \times T^*Y \times E$ , with  $f(x) = \pi_Y(\xi) = q(e)$ . Using this vector bundle, we write down the diagram

$$\begin{array}{ccc} \pi_Y^*E & \xrightarrow{\sigma_{D_E}} & \pi_Y^*E \\ a \uparrow & & a \uparrow \\ f^*(T^*Y \oplus E) & \xrightarrow{\widetilde{\sigma_{D_E}}} & f^*(T^*Y \oplus E) \\ b \downarrow & & b \downarrow \\ \pi_X^*(f^*E) & \xrightarrow{\sigma_{D_{f^*E}}} & \pi_X^*(f^*E), \end{array} \quad (15.4)$$

where for all  $(x, \xi, e) \in f^*(T^*Y \oplus E)$ ,

$$\begin{aligned} a(x, \xi, e) &:= (\xi, e) \\ b(x, \xi, e) &:= ((T_x f)^* \xi, x, e) \\ \widetilde{\sigma_{D_E}}(x, \xi, e) &:= (x, \sigma_{D_E}(\xi, e)). \end{aligned}$$

**Lemma 15.4.** *Diagram (15.4) commutes.*

*Proof.* The upper half of diagram (15.4) commutes by definition of the map  $\widetilde{\sigma_{D_E}}$ .

To prove commutativity of the lower half, let  $x \in X$ ,  $\varphi \in C^\infty(Y)$  and  $s \in \Gamma^\infty(E)$  be given. Then

$$\begin{aligned} \sigma_{D_{f^*E}}(b(x, d_{f(x)}\varphi, s(f(x)))) &= \sigma_{D_{f^*E}}((T_x f)^* d_{f(x)}\varphi, x, s(f(x))) \\ &= \sigma_{D_{f^*E}}(d_x(f^*\varphi), (f^*s)(x)) \\ &= (d_x(f^*\varphi), \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^d} (e^{-i\lambda f^*\varphi} D_{f^*E} e^{i\lambda f^*\varphi} f^*s)(x)) \\ &= (d_x(f^*\varphi), \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^d} (f^*(e^{-i\lambda\varphi} D_E e^{i\lambda\varphi} s)(x))) \\ &= (d_x(f^*\varphi), \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^d} (x, (e^{-i\lambda\varphi} D_E e^{i\lambda\varphi} s)(f(x)))) \\ &= ((T_x f)^* d_{f(x)}\varphi, x, \sigma_{D_E}(d_{f(x)}\varphi, s(f(x)))) \\ &= b(\widetilde{\sigma_{D_E}}(x, d_{f(x)}\varphi, s(f(x)))). \end{aligned}$$

□

Rather than diagram (15.4), we would prefer a diagram with a direct vector bundle homomorphism from  $\pi_Y^* E$  to  $\pi_X^*(f^* E)$  in it. It is however impossible to define such a map in general. The best we can do is to define it for each point  $x \in X$  separately: let

$$(b \circ a^{-1})_x : \pi_Y^* E|_{T_{f(x)}^* Y} \rightarrow \pi_X^*(f^* E)|_{T_x^* X}$$

be the map

$$(b \circ a^{-1})_x(\xi, e) = ((T_x f)^* \xi, e).$$

Using this map, we obtain the following statement, which is actually equivalent to Lemma 15.4.

**Corollary 15.5.** *For all  $x \in X$ , the following diagram commutes:*

$$\begin{array}{ccc} \pi_Y^* E|_{T_{f(x)}^* Y} & \xrightarrow{\sigma_{D_E}|_{T_{f(x)}^* Y}} & \pi_Y^* E|_{T_{f(x)}^* Y} \\ \downarrow (b \circ a^{-1})_x & & \downarrow (b \circ a^{-1})_x \\ \pi_X^*(f^* E)|_{T_x^* X} & \xrightarrow{\sigma_{D_{f^*E}}|_{T_x^* X}} & \pi_X^*(f^* E)|_{T_x^* X}. \end{array}$$

One last remark that we will use later, is that the maps  $(b \circ a^{-1})_x$  are injective if  $T_x f$  is surjective. So if  $f$  is a submersion, all  $(b \circ a^{-1})_x$  are injective.

### 15.3 The principal symbols of $\mathcal{D}_M^{L^{2\omega}}$ and $\tilde{\mathcal{D}}_M^{L^{2\omega}}$ .

Let  $g^N$  and  $g^M$  be the Riemannian metrics on  $N$  and  $M$ , respectively, induced by the  $\text{Spin}^c$ -structures  $P^N$  and  $P^M$ . We use the same notation for the map  $g^M : TM \rightarrow T^*M$  given by  $v \mapsto g^M(v, -)$ , and similarly for  $g^N$ . The Dirac operators  $\mathcal{D}_M^{L^{2\omega}}$  and  $\mathcal{D}_N^{L^{2\nu}}$  have principal symbols

$$\begin{aligned}\sigma_{\mathcal{D}_M^{L^{2\omega}}} : \pi_M^* \mathcal{S}^M &\rightarrow \pi_M^* \mathcal{S}^M; \\ \sigma_{\mathcal{D}_N^{L^{2\nu}}} : \pi_N^* \mathcal{S}^N &\rightarrow \pi_N^* \mathcal{S}^N,\end{aligned}$$

given by the Clifford action (3.10):

$$\begin{aligned}\sigma_{\mathcal{D}_M^{L^{2\omega}}}(\xi, s^M) &= (\xi, c_{TM}(i(g^M)^{-1}(\xi))s^M); \\ \sigma_{\mathcal{D}_N^{L^{2\nu}}}(\eta, s^N) &= (\eta, c_{TN}(i(g^N)^{-1}(\eta))s^N),\end{aligned}\tag{15.5}$$

for  $m \in M$ ,  $\xi \in T_m^*M$ ,  $s^M \in \mathcal{S}_m^M$  and  $n \in N$ ,  $\eta \in T_n^*N$ ,  $s^N \in \mathcal{S}_n^N$ .

To determine the principal symbol of  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$ , we need the following basic fact:

**Lemma 15.6.** *The principal symbol of the operator  $\mathcal{D}_{G,K}$  on the trivial bundle  $G \times \Delta_{d_p} \rightarrow G$  is given by*

$$\sigma_{\mathcal{D}_{G,K}}(g, \xi, \delta_p) = (g, \xi, c_p(i\xi_{p^*})\delta_p),$$

for  $g \in G$ ,  $\xi \in \mathfrak{g}^*$  and  $\delta_p \in \Delta_{d_p}$ . Here  $\xi_{p^*}$  is the component of  $\xi$  in  $\mathfrak{p}^* \cong \mathfrak{k}^0$  according to  $\mathfrak{g}^* = \mathfrak{p}^0 \oplus \mathfrak{k}^0$ , and we identify  $\mathfrak{p}^*$  with  $\mathfrak{p}$ , and  $\mathfrak{p}$  with  $\mathbb{R}^{d_p}$ , using a  $B$ -orthonormal basis  $\{X_1, \dots, X_{d_p}\}$  of  $\mathfrak{p}$ .

*Proof.* Let  $g \in G$ ,  $f \in C^\infty(G)$  and  $\tau \in C^\infty(G, \Delta_{d_p})$  be given. Then

$$\begin{aligned}\sigma_{\mathcal{D}_{G,K}}(d_g f, \tau(g)) &= (d_g f, \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} (e^{-i\lambda f} \mathcal{D}_{G,K}(e^{i\lambda f} \tau))(g)) \\ &= (d_g f, \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} (e^{-i\lambda f} \sum_j c_p(X_j) X_j(e^{i\lambda f} \tau))(g)).\end{aligned}$$

This expression equals

$$\left( d_g f, \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \left( \sum_j c_p(X_j) (i\lambda X_j(f) \tau + X_j(\tau)) \right) (g) \right) = \left( d_g f, i \sum_j c_p(X_j) \langle d_g f, T_{e_l g}(X_j) \rangle \tau(g) \right).$$

Hence for all  $\xi \in \mathfrak{g}^*$ ,  $\delta_p \in \Delta_{d_p}$ , we have

$$\begin{aligned}\sigma_{\mathcal{D}_{G,K}}(g, \xi, \delta_p) &= \left( g, \xi, i \sum_j c_p(\langle \xi, X_j \rangle X_j) \delta_p \right) \\ &= (g, \xi, c_p(i\xi_p) \delta_p),\end{aligned}$$

since  $\{X_j\}$  is a basis of  $\mathfrak{p}$ , orthonormal with respect to the Killing form.  $\square$

We are now ready to prove that  $\mathcal{D}_M^{L^{2\omega}}$  and  $\tilde{\mathcal{D}}_M^{L^{2\omega}}$  have the same principal symbol, and hence define the same class in  $K$ -homology. This will conclude the proof of Proposition 14.6, which was the remaining step in the proof of Theorem 14.5. As we saw in Section 14.3, the latter theorem implies Theorem 6.13, which is our second main result.

**Proposition 15.7.** *The following diagram commutes:*

$$\begin{array}{ccc}
 \pi_M^* \mathcal{S}^M & \xrightarrow{\sigma_{\mathcal{D}_M^{L^{2\omega}}}} & \pi_M^* \mathcal{S}^M \\
 \downarrow \cong & & \downarrow \cong \\
 \pi_M^*(((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K) & \xrightarrow{\sigma_{\tilde{\mathcal{D}}_M^{L^{2\omega}}}} & \pi_M^*(((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K) \\
 \uparrow a & & \uparrow a \\
 p^*(T^*M \oplus ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K) & \xrightarrow{\widetilde{\sigma_{\tilde{\mathcal{D}}_M^{L^{2\omega}}}}} & p^*(T^*M \oplus ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K) \\
 \downarrow b & & \downarrow b \\
 \pi_{G \times N}^*(p^*(((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K)) & \longrightarrow & \pi_{G \times N}^*(p^*(((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K)) \\
 \downarrow h \cong & & \downarrow h \cong \\
 \pi_{G \times N}^*((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N) & \xrightarrow{\sigma_{\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^{L^{2v}}}} & \pi_{G \times N}^*((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N) \\
 \downarrow \theta \cong & & \downarrow \theta \cong \\
 \pi_G^*(G \times \Delta_{d_p}) \boxtimes \pi_N^* \mathcal{S}^N & \xrightarrow{\sigma_{\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \sigma_{\mathcal{D}_N^{L^{2v}}}}} & \pi_G^*(G \times \Delta_{d_p}) \boxtimes \pi_N^* \mathcal{S}^N.
 \end{array} \tag{15.6}$$

Here the isomorphism  $h$  is induced by the general isomorphism  $p^*(E/H) \cong E$ , as defined in (8.8). The fourth horizontal map from the top is just defined as the composition  $h^{-1} \circ (\sigma_{\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^{L^{2v}}}) \circ h$ , i.e. by commutativity of the second square from the bottom.

*Proof.* It follows from Lemma 15.3 that the bottom square of (15.6) commutes. Note that

$$(\mathcal{D}_{G,K} \otimes 1 + 1 \otimes \mathcal{D}_N^{L^{2v}}) p^* s = p^* (\tilde{\mathcal{D}}_M^{L^{2\omega}} s)$$

for all  $s \in \Gamma^\infty(((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K)$  (see the sketch of the proof of Proposition 8.6). We can therefore apply Lemma 15.4 to see that the second and third squares in (15.6) from the top commute as well. We will first show that the outside of diagram (15.6) commutes, and then deduce commutativity of the top subdiagram.

Let  $g \in G$ ,  $n \in N$ ,  $\eta \in T_n^* N$ ,  $\xi \in \mathfrak{p}^*$ ,  $p^N \in P^N$ ,  $\delta_p \in \Delta_{d_p}$  and  $\delta_N \in \Delta_{d_N}$  be given. Then we have the element

$$((g, n), [g, \eta, \xi], [(g, \delta_p) \otimes [p^N, \delta_N]]) \in p^*(T^*M \oplus ((G \times \Delta_{d_p}) \boxtimes \mathcal{S}^N)/K). \tag{15.7}$$

Here we have used Proposition 12.1 and Lemma 12.2. Applying the map  $a$  and the (inverse of the) isomorphism in the upper left corner of (15.6) to this element, we obtain

$$\begin{aligned}
 & ([g, \eta, \xi], [[g, n, e_{\text{Spin}^c(\mathfrak{p})}], [g, p^N], \delta_p \otimes \delta_N]) \\
 & \in \pi_M^*(P_M^{G/K} \times_{U(1)} (G \times_K P^N) \times_H \Delta_{d_p} \otimes \Delta_{d_N}) \cong \pi_M^* \mathcal{S}^M. \tag{15.8}
 \end{aligned}$$

Here  $e_{\text{Spin}^c(\mathfrak{p})}$  is the identity element of  $\text{Spin}^c(\mathfrak{p})$ .

Let  $\zeta \in (\mathbb{R}^{d_N})^*$  be the covector such that  $\eta \in T^*N$  corresponds to  $[p^N, \zeta] \in P^N \times_{\text{Spin}^c(d_N)} (\mathbb{R}^{d_N})^*$ . Then  $\sigma_{\tilde{\mathcal{D}}_M^{L^{2\omega}}}$  applied to (15.8) gives

$$([g, \eta, \xi], [[g, n, e_{\text{Spin}^c(\mathfrak{p})}], [g, p^N], c_{\mathfrak{p} \oplus \mathbb{R}^{d_N}}(i\xi, i\zeta)(\delta_{\mathfrak{p}} \otimes \delta_N)]),$$

where we identify  $(\mathbb{R}^{d_N})^* \cong \mathbb{R}^{d_N}$  using the standard Euclidean metric, and  $\mathfrak{p}^* \cong \mathfrak{p}$  using the Killing form. By definition of the Clifford modules  $\Delta_k$  (see e.g. [22], page 13), this equals

$$([g, \eta, \xi], [[g, n, e_{\text{Spin}^c(\mathfrak{p})}], [g, p^N], c_{\mathfrak{p}}(i\xi)\delta_{\mathfrak{p}} \otimes \delta_N + \delta_{\mathfrak{p}} \otimes c_{\mathbb{R}^{d_N}}(i\zeta)\delta_{d_N}]).$$

(This is the central step in the proof of Proposition 14.6.)

The image of the latter element under the maps  $\theta \circ h \circ (b \circ a^{-1})_{(g,n)}$  is

$$((g, \xi), (g, c_{\mathfrak{p}}(i\xi)\delta_{\mathfrak{p}})) \otimes (\eta, [p^N, \delta_N]) + ((g, \xi), (g, \delta_{\mathfrak{p}})) \otimes (\eta, [p^N, c_{\mathbb{R}^{d_N}}(i\zeta)\delta_{d_N}]),$$

which by Lemma 15.6 equals the image under the map

$$(\sigma_{\mathcal{D}_{G,K}} \otimes 1 + 1 \otimes \sigma_{\mathcal{D}_N^{L^{2\nu}}}) \circ \theta \circ h \circ b$$

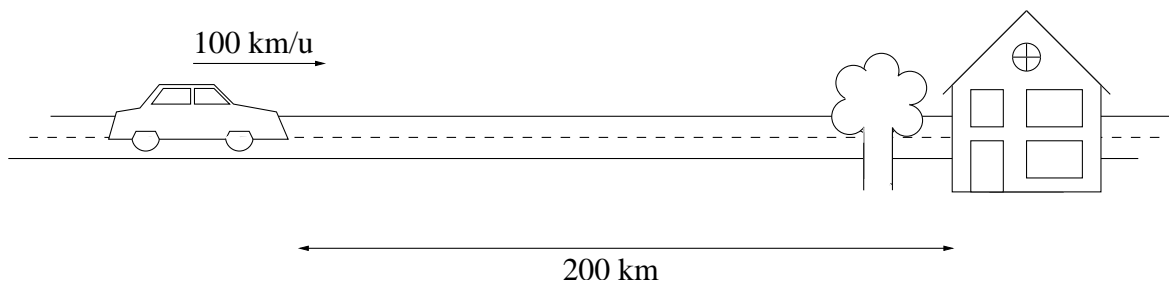
of (15.7). Therefore, the outside of diagram (15.6) commutes.

Now note that for all  $(g, n) \in G \times N$ , the composition  $\theta \circ h \circ (b \circ a^{-1})_{(g,n)}$  is injective, because  $p$  is a submersion (see the remark after Corollary 15.5). This fact, together with commutativity of the outside of diagram (15.6), implies that the top part of (15.6) commutes as well.  $\square$

# Samenvatting in het Nederlands

Een van de nadelen van het promoveren in de wiskunde is dat je nooit over je werk kan praten met mensen die niet weten wat bijvoorbeeld de  $K$ -theorie van een  $C^*$ -algebra is (ze weten *niet* wat ze missen<sup>2</sup>). Aan de andere kant geeft dat je werk ook wel een soort mysterieuze charme (toch...?). In ieder geval ga ik in deze samenvatting toch proberen om iets over mijn onderzoek te zeggen dat ook begrijpelijk is voor mensen die geen wiskunde gestudeerd hebben. Ik ben er al vier jaar over aan het nadenken hoe ik dat het beste kan aanpakken, en uiteindelijk heb ik besloten dat ik de titel van mijn proefschrift ga uitleggen aan de hand van een voorbeeld.

De Nederlandse vertaling van de titel van mijn proefschrift is “Kwantisatie<sup>3</sup> commuteert met reductie voor cocompacte Hamiltonse groepsacties.” Een redelijk cryptische zin voor de meeste mensen. Het belangrijkste deel van die titel is het eerste stuk: “kwantisatie commuteert met reductie”. Ik zal die woorden uitleggen aan de hand van een auto op de snelweg, zoals in Figuur 1.



Figuur 1: Een auto op de snelweg

## Kwantisatie

Eerst het woord ‘kwantisatie’. Dat betekent dat je van de normale, klassieke beschrijving van een situatie de kwantummechanische beschrijving ervan maakt.

Wat betekent dat in het geval van de auto? De klassieke beschrijving van de situatie is wat we allemaal gewend zijn. Stel, je rijdt in een auto en je vraagt je af hoe laat je thuis zal zijn. Als je dan (zoals in Figuur 1) weet dat je 200 km van huis bent, en je 100 km per uur rijdt, dan weet je ook dat je over twee uur thuis bent. Je kan natuurlijk in de tussentijd in de file

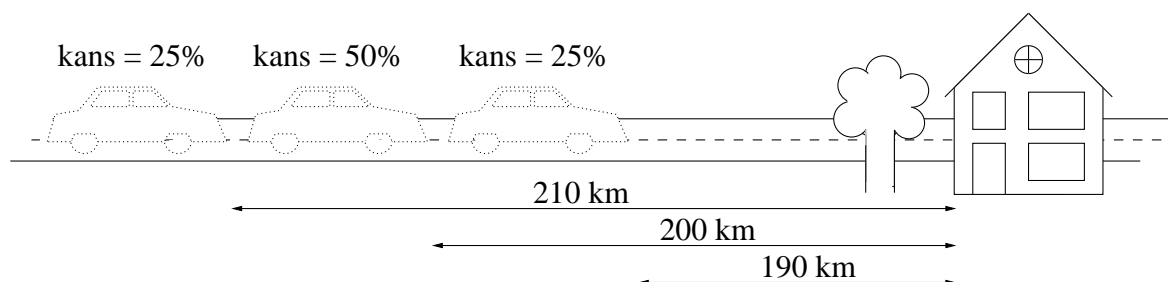
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<sup>2</sup>Zie paragraaf 4.2.

<sup>3</sup>Dat schrijf je sinds 1996 inderdaad met ‘kw’.

komen te staan, of haast krijgen en 150 km per uur gaan rijden, maar dat zou het verhaal een beetje verpesten. De twee dingen die je moet weten zijn dus *waar je bent* (hoe ver van huis bijvoorbeeld) en *hoe hard je gaat*. Die twee stukjes informatie, plaats en snelheid, noemen we de klassieke beschrijving<sup>4</sup> van de situatie.

De kwantummechanica is de natuurkunde van de erg kleine dingen. Daarbij gaat het er volkomen anders aan toe dan je gewend bent. Het belangrijkste punt in de kwantummechanica is dat je niet meer zeker weet *waar* iets precies is, maar dat je alleen de *kans* weet dat iets hier of daar is. Als je een auto op de snelweg op een kwantummechanische manier beschrijft, dan weet je niet meer of je 190, 200 of 210 km van huis bent, maar alleen de *kans* dat je nog zo ver moet rijden, zoals bijvoorbeeld in Figuur 2. In dit voorbeeld kan de auto op drie plaatsen zijn,



Figuur 2: Een kwantum-auto

maar het kunnen er net zo goed twee, zeven of zelfs oneindig veel zijn.

Dat is natuurlijk onzin, in het echt weet je best waar je bent. Dit gaat ook alleen maar op voor auto's die kleiner zijn dan zeg 0,0000001 mm. Dus zelfs met een Nissan Micra of een Smart merk je er niets van.

De snelheid van de auto mogen we nu vergeten. Als je de kansverdeling<sup>5</sup> weet van de plaats van een auto, dan blijkt je via een wiskundig trucje<sup>6</sup> ook de kansverdeling van zijn snelheid te kunnen bepalen, maar dat laten we nu even zitten.

Wat betekent het woord 'kwantisatie' nu? Dat betekent dat je de klassieke beschrijving neemt, de verzameling van *alle mogelijke* plaatsen en snelheden van de auto (zoals in Figuur 1), en die vervangt door de kwantummechanische beschrijving, de verzameling van *alle mogelijke* kansverdelingen van de plaats van de auto (zoals in Figuur 2).<sup>7</sup>

<sup>4</sup>In dit proefschrift komt vaak de term 'symplectische variëteit' ('symplectic manifold' in het Engels, zie Definition 2.1) voor. Dat is min of meer de verzameling van alle mogelijke plaatsen en snelheden van een auto, een knikker of wat dan ook. Dat een symplectische variëteit meestal  $(M, \omega)$  heet betekent trouwens niet dat  $M$  voor de plaats staat en  $\omega$  voor de snelheid. Plaats en snelheid zitten allebei in die  $M$ , en  $\omega$  is iets dat je kan gebruiken om te bepalen hoe die auto of die knikker verder gaat bewegen.

<sup>5</sup>Als ik het in dit proefschrift over een 'Hilbertruimte' ('Hilbert space') heb, dan is dat min of meer de verzameling van alle mogelijke kansverdelingen van de plaats van een auto, een knikker, of iets anders.

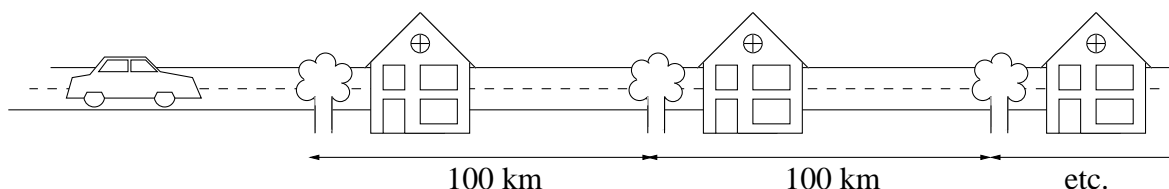
<sup>6</sup>de Fourier-transformatie

<sup>7</sup>Er zijn verschillende wiskundige definities van kwantisatie. Zie Definitions 3.15, 3.17, 3.20, 3.30, 6.1 en 6.2. Degene die ik gebruikt heb zijn de meest algemene, Definitions 6.1 en 6.2.



## Reductie

Nu het woord ‘reductie’. Dat heeft alles te maken met *symmetrie*. Een gezicht is bijvoorbeeld (bijna) spiegelsymmetrisch, en een appel (bijna) rotatiesymmetrisch. In het voorbeeld van de auto kijken we naar een ander soort symmetrie. Stel dat je door een saai, symmetrisch polder-landschap rijdt, met precies om de 100 km een boom en een huis (zie Figuur 3). Dat landschap

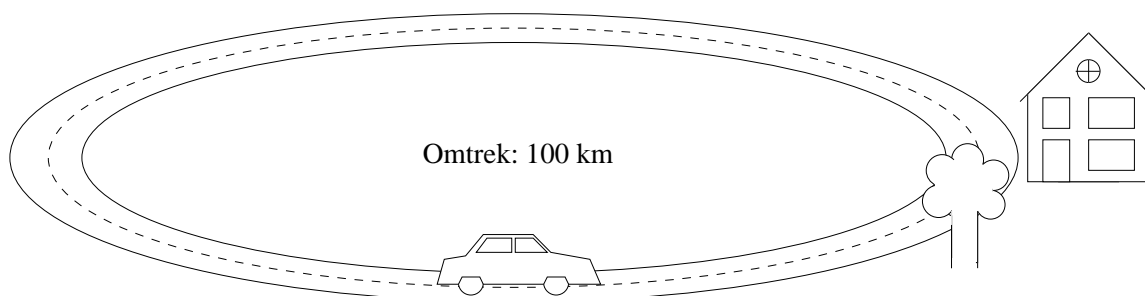


Figuur 3: Een klassieke auto in een symmetrisch landschap

blijft hetzelfde als je het 100 km opschuift. Met andere woorden: 100 km verschuiven is een *symmetrie*<sup>8</sup> van het landschap. Als alle bomen en huizen er hetzelfde uitzien tenminste, maar dat nemen we even aan.

Als je je nu niet afvraagt wanneer je bij *jouw* huis bent, maar wanneer je bij *een* huis bent, dan hoeft je niet meer te weten waar je precies op de weg zit, maar alleen hoe ver je van het dichtstbijzijnde huis bent. Het maakt dan niet uit of je 100 km verderop zit, of 200 km, etc.

Nu maken we even een denkstap. We zijn allemaal wel eens verdwaald geweest, en dan vraag je je soms af “Ben ik hier niet al eerder langs gereden?” Dat vraagt de automobilist in Figuur 3 zich ook af. Hij weet niet of alle bomen en huizen er hetzelfde uitzien, of dat hij in een rondje aan het rijden is, zoals in Figuur 4. Hij weet natuurlijk wel of hij naar links moet



Figuur 4: De reductie: een ronde weg

sturen of rechtdoor rijdt, maar op een ronde weg van 100 km merk je het verschil toch bijna niet. Figuur 4 heet de (*klassieke*) *reductie* van Figuur 3. Of, om preciezer te zijn, de verzameling van *alle mogelijke* plaatsen en snelheden van een auto op de ronde weg in Figuur 4 is de reductie

<sup>8</sup>De termen ‘groep’ (‘group’), ‘groepsactie’ (‘group action’) of zelfs ‘Hamiltonse groepsactie’ (‘Hamiltonian group action’, Definition 2.6) in dit proefschrift slaan allemaal op zulke symmetrieën. In dit voorbeeld is de *groep* de verzameling van alle gehele getallen  $n = \dots, -1, 0, 1, 2, 3, \dots$ , en de *groepsactie* is het verschuiven van de weg over  $n$  maal 100 km. Deze groepsactie blijkt Hamiltons te zijn.

van de verzameling van *alle mogelijke* plaatsen en snelheden van een auto op de symmetrische weg in Figuur 3.

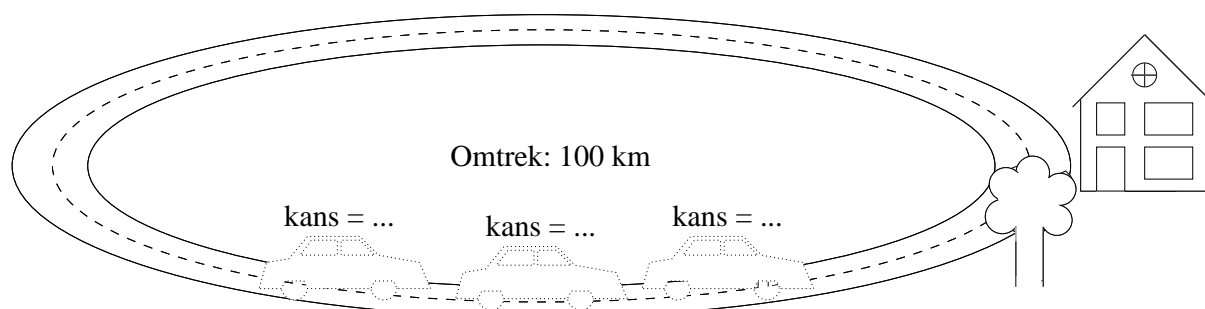
Als er iets symmetrisch aan de hand is, dan kun je vaak net zo goed naar een kleinere situatie kijken, zoals de weg in Figuur 4 kleiner is (namelijk 100 km lang) dan de weg in Figuur 3 (oneindig lang). Die kleinere situatie heet dan de reductie<sup>9</sup> van de symmetrische situatie. Het is vaak makkelijker om met de reductie te werken dan met de grote situatie, hoewel dat niet direct uit dit voorbeeld blijkt.

## Commuteert kwantisatie met reductie?

Nu komt alles samen dat we tot zover gezien hebben. Dat kan wat veel informatie tegelijk zijn, dus dit is even een moment om goed op te letten.

Zoals ik al zei is centrale thema van mijn proefschrift de zin “Kwantisatie commuteert met reductie”. Die betekent dat eerst de klassieke reductie nemen, en daarvan de kwantisatie, hetzelfde oplevert als eerste de kwantisatie nemen, en daarvan de kwantum-reductie.<sup>10</sup>

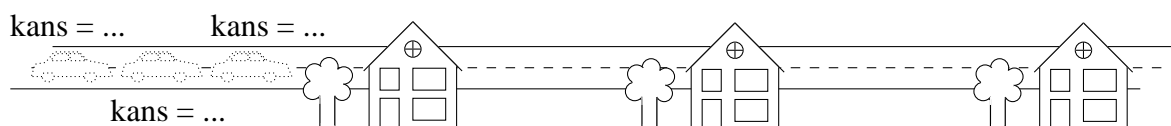
De reductie van Figuur 3 is Figuur 4. De kwantisatie van die reductie is de kwantummechanische versie van Figuur 4, die in Figuur 5 uitgebeeld is. Hier bedoel ik eigenlijk weer de



Figuur 5: De kwantisatie van de reductie

verzameling van alle mogelijke kansverdelingen van de plaats van de auto op de ronde weg.

Dit willen we vergelijken met de (kwantum-)reductie van de kwantisatie van Figuur 3. Die kwantisatie ziet eruit als Figuur 6.

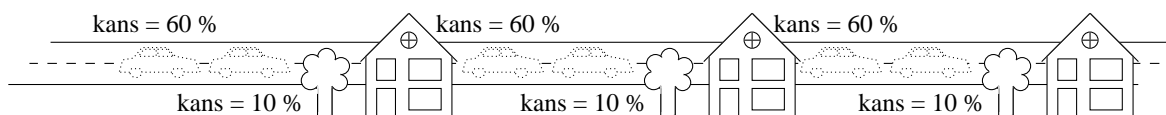


Figuur 6: Een kwantum-auto in een symmetrisch landschap

<sup>9</sup>Zie Definition 2.17.

<sup>10</sup>Op de voorkant van dit proefschrift staat de afkorting  $[Q, R] = 0$  van de zin “Kwantisatie commuteert met reductie”. In die afkorting staat  $Q$  voor kwantisatie (‘quantisation’),  $R$  voor reductie, en  $[Q, R]$  voor het ‘verschil’ tussen eerst de reductie nemen en dan de kwantisatie en eerst de kwantisatie nemen en daarna de reductie. Dat verschil is niet echt goed gedefinieerd, dus  $[Q, R] = 0$  is een symbolische afkorting, en niet een echte formule.

Maar wat is daar de reductie van? Dat is een moeilijke vraag. Je wil in ieder geval dat die reductie hetzelfde is als Figuur 5, zodat kwantisatie inderdaad met reductie commuteert. Maar de standaardmanier<sup>11</sup> om de reductie van Figuur 6 te definiëren is om de verzameling te nemen van alle kansverdelingen die niet veranderen als je ze verschuift over 100 km. Een voorbeeld van zo'n kansverdeling staat in Figuur 7. Dat is helaas een onzinnige kansverdeling.



Figuur 7: De reductie van de kwantisatie?

Alle kansen samen zouden namelijk precies 1 moeten zijn, maar in Figuur 7 zijn alle kansen samen gelijk aan

$$60\% + 10\% + 60\% + 10\% + 60\% + 10\% + \dots,$$

en daar komt niet 1 uit. (Er komt zelfs ‘oneindig’ uit, wat al helemaal nergens op slaat.)

Dus commuteert kwantisatie nu met reductie? In dit voorbeeld weten we niet eens wat de reductie van de kwantisatie is, dus we kunnen de vraag überhaupt niet goed formuleren... Dat probleem wordt veroorzaakt doordat de weg die we bekijken oneindig uitgestrekt is, waardoor een goede kansverdeling nooit hetzelfde kan blijven als je hem 100 km opschuift, zoals we net zagen.

## Compact en niet-compact

Iets dat oneindig uitgestrekt is, zoals de weg in Figuur 3, noemen we in de wiskunde *niet-compact*. Voorbeelden van andere niet-compacte dingen zijn lijnen, vlakken en oneindig lange cilinders. Wél compact zijn bijvoorbeeld cirkels (zoals de weg in Figuur 4), boloppervlakken en oppervlakken van autobanden, want die zijn begrensd.<sup>12</sup>

In de jaren '80 en '90 is er een hoop (wiskundig) onderzoek gedaan naar de vraag of kwantisatie commuteert met reductie, maar alleen als alles compact is. (En dan blijkt het antwoord “Ja” te zijn.) Omdat je in het niet-compacte geval problemen krijgt zoals ik hierboven uitlegde, was daar nog nooit naar gekeken. Mijn promotor Klaas Landsman heeft een manier gevonden om ook in niet-compacte situaties de vraag of kwantisatie commuteert met reductie op een wiskundig precieze manier te stellen.<sup>13</sup> De afgelopen 4 jaar heb ik geprobeerd om die vraag voor zo veel mogelijk situaties te beantwoorden. In de situaties die ik bekeken heb, is het antwoord weer “Ja”.<sup>14</sup>

<sup>11</sup>Zie (3.15).

<sup>12</sup>Ik wek hier misschien de indruk dat ‘compact’ hetzelfde betekent als ‘begrensd’, maar dat is niet helemaal zo. Een begrensd lijnstuk waarvan de eindpunten niet meedoen is bijvoorbeeld niet compact. Als de eindpunten wel meedoen is zo'n lijnstuk wel compact. Het cruciale verschil is dat een continue functie op een lijnstuk *met* eindpunten altijd een maximale en minimale waarde aanneemt, terwijl dat niet zo is voor een lijnstuk *zonder* eindpunten. Denk bijvoorbeeld aan de functie  $f(x) = \frac{1}{x}$  op het lijnstuk  $]0, 1[$ , dat bestaat uit alle getallen die groter zijn dan 0 en kleiner dan 1.

<sup>13</sup>Zie Conjecture 6.4. (‘Conjecture’ betekent ‘vermoeden’.)

<sup>14</sup>Zie Theorems 6.5 en 6.13. (‘Theorem’ betekent ‘stelling’.)

Ik heb dus naar niet-compacte snelwegen gekeken, zoals in Figuur 3, maar alleen als ze zo symmetrisch waren dat hun reductie compact was, zoals de ronde, begrensde weg in Figuur 4. Dat is de betekenis van het woord ‘cocompact’ in de titel van mijn proefschrift.

Tot slot moet ik bekennen dat het voorbeeld in deze samenvatting niet in mijn proefschrift past, omdat de reductie in Figuur 4 toch eigenlijk niet compact is. De oorzaak daarvan is dat een auto op een ronde weg wel elke snelheid kan hebben die je wil. (Dit is nu niet alleen een wiskundige utopie, maar meer een algemeen mannelijke. Waarmee ik niet wil beweren dat vrouwen geen wiskunde kunnen doen, of niet hard zouden willen rijden natuurlijk.) Het snelheids-gedeelte van Figuur 4 is daardoor wel oneindig uitgestrekt, oftewel niet compact. In Section 11.6 bekijk ik een variant van dit voorbeeld waarbij het ook niet uitmaakt of je bijvoorbeeld 80 km per uur rijdt of 180, of 280, etc. Dat heeft niets meer met de realiteit te maken, maar dan commuteert kwantisatie wel mooi met reductie.<sup>15</sup>

## Maar wat *heb* je daar nou aan?

Als iemand iets over wiskunde schrijft of vertelt, dan raak ik meestal snel mijn interesse kwijt als ik niet snap *waarom* je naar de wiskunde zou willen kijken waar het over gaat. Daar wordt vaak weinig aandacht aan besteed, omdat het meestal moeilijk uit te leggen is. Dat geldt ook voor mijn proefschrift, maar ik wil toch een paar redenen noemen waarom je het interessant of nuttig kan vinden dat kwantisatie commuteert met reductie.

Ten eerste is het een test voor de definities van kwantisatie en reductie. Als kwantisatie niet commuteert met de reductie, dan is er (vind ik) iets mis met de definitie van kwantisatie en/of reductie. Mijn begeleider Klaas Landsman heeft definities bedacht van kwantisatie en (kwantum-)reductie, en het is dus een goed teken dat met die definities kwantisatie en reductie inderdaad met elkaar commuteren, in de gevallen die ik bekeken heb.

Ten tweede is het vaak niet makkelijk om de kwantisatie te bepalen van een klassieke reductie. Maar als kwantisatie commuteert met reductie, dan kun je, in plaats van die klassieke reductie te kwantiseren, net zo goed de hele situatie kwantiseren (wat makkelijker is), en daarvan de reductie nemen (wat ook te doen moet zijn).

De derde reden is voor mij de belangrijkste. Die reden is dat “kwantisatie commuteert met reductie” een verband aangeeft tussen de *wiskunde* achter de klassieke mechanica en de *wiskunde* achter de kwantummechanica. En de stukjes wiskunde die ik het mooist vind zijn de stukjes die een verband aangeven tussen dingen die op het eerste gezicht totaal verschillend lijken.

De stellingen in dit proefschrift zijn zo abstract dat natuurkundigen er (nog...) niets aan hebben. Maar ze geven wel een verband aan tussen de wiskunde achter de klassieke mechanica, die *symplectische meetkunde* heet, en de wiskunde achter de kwantummechanica, die *representatietheorie* heet, of in mijn geval *K-theorie*. Die vakgebieden lijken niets met elkaar te maken te hebben, als je niet weet dat kwantisatie commuteert met reductie. Dat er wél een verband is tussen die onderwerpen is niet alleen mooi, maar zorgt er ook voor dat we ze allebei beter gaan begrijpen. En daar houden wij van, van dingen begrijpen.

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<sup>15</sup>Zie diagram (11.14).

# Dankwoord/ Acknowledgements

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# Notation

## Topological spaces

For any topological space  $X$ , and any (continuous) vector bundle  $E$  over  $X$ ,

- $C(X)$ : the space of continuous functions on  $X$ ;
- $C_c(X)$ : the space of compactly supported continuous functions on  $X$ ;
- $\Gamma(E) = \Gamma(M, E)$ : the space of continuous sections of  $E$ ;
- $\Gamma_c(E) = \Gamma_c(M, E)$ : the space of compactly supported continuous sections of  $E$ ;
- $E \boxtimes F$ : if  $F \rightarrow Y$  is another vector bundle, the exterior product vector bundle over  $X \times Y$ ;
- $L^2(X), L^2(X, E)$ : if  $X$  is equipped with a measure, the Hilbert space of  $L^2$ -functions on  $X$  and the Hilbert space of  $L^2$ -sections of a Hermitian vector bundle  $E$  over  $X$ ;
- $X^+$ : the one-point compactification of  $X$ , if  $X$  is locally compact;
- $\text{pt}$ : the one-point space.

## Smooth manifolds

For any smooth manifold  $M$ , and any (smooth) vector bundle  $E$  over  $M$ ,

- $C^\infty(M)$ : the space of smooth functions on  $M$ ;
- $C_c^\infty(M)$ : the space of compactly supported smooth functions on  $M$ ;
- $\Gamma^\infty(E) = \Gamma^\infty(M, E)$ : the space of smooth sections of  $E$ ;
- $\Gamma_c^\infty(E) = \Gamma_c^\infty(M, E)$ : the space of compactly supported smooth sections of  $E$ ;
- $\Omega^k(M; E)$ : the space of smooth sections of  $\bigwedge^k T^*M \otimes E \rightarrow M$ ;
- $\Omega^{p,q}(M; E)$ : the space of smooth sections of  $\bigwedge^{p,q} T^*M \otimes E \rightarrow M$ , if  $M$  is equipped with an almost complex structure;
- $\mathfrak{X}(M)$ : the space of smooth vector fields on  $M$ ;
- $i_v$ : contraction of differential forms by the vector field  $v$ ;



- $R_{\nabla}$ : the curvature of a connection  $\nabla$  on  $E$ ;
- $\sigma_D$ : the principal symbol of a (pseudo-)differential operator  $D$  on  $E$ .

### **Lie groups, Lie algebras and representations**

- $\mathfrak{g}, \mathfrak{h}$ : the Lie algebras of Lie groups  $G, H$  etc.;
- $B$ : the Killing form on a Lie algebra;
- $[V : W]$ : the multiplicity of a representation  $W$  in a (finite-dimensional) representation  $V$ ;
- $V_{\lambda}$ : the irreducible representation of a compact Lie group with highest weight  $\lambda \in \Lambda_+$ ;
- $T^{\text{reg}}$ : the regular elements of a torus  $T$ , i.e. the set  $\{\exp X; X \in \mathfrak{t}, \langle \alpha, X \rangle \notin 2\pi i\mathbb{Z} \text{ for all roots } \alpha\}$ ;
- $X^G$ : for  $X$  a set equipped with an action by a group  $G$ , the set of fixed points of the action;
- $\mathcal{L}_X$ : for  $X$  in the Lie algebra of a Lie group acting on a smooth manifold, the Lie derivative of differential forms, with respect to  $X$ ;
- $V^0$ : for  $V$  a subspace of a vector space  $W$ , the annihilator  $\{\xi \in W^*; \xi|_V = 0\}$ .

# Curriculum vitae

De auteur is geboren op 1 november 1977 in Den Bosch. Daar bezocht hij vanaf 1990 het Stedelijk Gymnasium, dat hem in 1996 zijn VWO-diploma gaf.

Vervolgens begon hij in september 1996 aan het Tweelingprogramma Wis- en Natuurkunde van de Universiteit Utrecht. Hij onderbrak zijn studie van september 1999 tot september 2000 voor een jaar bestuur van studentenvereniging Biton. In maart 2003 studeerde hij (cum laude) af in de wiskunde bij dr. E. P. van den Ban, op de scriptie “A Lefschetz fixed point theorem and the Weyl character formula”. In december van dat jaar kreeg hij ook zijn doctoraaldiploma natuurkunde.

In april 2003 trad de auteur in dienst van de Universiteit van Amsterdam, als promovendus van prof. N. P. Landsman in de mathematische fysica. In september 2004 verruilden prof. Landsman en zijn onderzoeksgroep de Universiteit van Amsterdam voor de Radboud Universiteit in Nijmegen. Daar zette de auteur zijn promotieonderzoek voort tot augustus 2007. Het resultaat van dat onderzoek is beschreven in dit proefschrift.

Sinds september 2007 werkt de auteur als projectmedewerker bij TNO Bouw en Ondergrond.